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A Set-Complete Domain Construction for Order-Sorted Set-Valued Features

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Abstract

Order-Sorted Feature ($\mathcal{OSF}$) Logic is a formalism of structured objects and types called "$\mathcal{OSF}$ terms." These are essentially labelled graphs. In such a graph, a node is labelled by a symbol called a "sort" and an arrow is labelled by a symbol called a "feature." Sorts denote sets of values and are partially ordered by a relation denoting set inclusion. A feature symbol denotes a function from the set denoted by its source-node's sort to the set denoted by its sink-node's sort. Description Logic (DL) is the formalism used by the W3C's official Semantic Web "Web Ontology Language" $\mathcal{OWL}$. It also uses partially ordered set-denoting symbols called "concepts." In it, pairs of concepts can be specified as the domain and range of symbols called "roles." A role is a symbol denoting a binary relation; *viz.*, a subset of the Cartesian product of its domain and range. Thus, in order to express a DL role in $\mathcal{OSF}$ Logic, one can use a set-valued feature. This may be done at the syntax level by introducing a new kind of sort: a "powersort," so to speak. Such a powersort is written using a "**set-of**" construct applicable to a sort whereby "**set-of**(s)" denotes the powerset of the sort s. This paper presents a set completion construction of a lower-semilattice ordering of sorts.

Keywords: Semantic Web; Description Logic; Order-Sorted Feature Logic; Formal Semantics; Concepual Role; Set-Valued Feature; Powersort.

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1 Introduction

The first comprehensive account of the formal basis for Description Logic was initially proposed as a logic of attributive concepts [15]. The common aspect of all such logics is that they describe so-called *concepts*. These "concepts" denote sets of elements of the universe in which they take their meaning. They are partially ordered with an "*is-a*" relation that denotes set inclusion—thus defining a "conceptual taxonomy." Description Logic is an elaboration of such conceptual taxonomies allowing expressing various properties on concepts. Such a property, in particular, is that of a *role* [6]. A role $r : a \times b$ with *domain* a (a concept) and *range* b (a concept) is simply defined as a binary relation, a subset of the Cartesian product of the set denoted by the concepts a and b; *i.e.*, $[\![r]\!] \subseteq [\![a]\!] \times [\![b]\!]$ (where "$[\![x]\!]$" stands for "*the formal denotation of* x"). For example, given a concept `person` denoting a set of persons, and a concept `activity` denoting a set of activities, we can define the role `hasHobby`, with domain `person` and range `activity`. It is meant to denote the set of all pairs of persons that have hobbies and their hobby activities (*i.e.*, $[\![\mathtt{hasHobby}]\!] \subseteq [\![\mathtt{person}]\!] \times [\![\mathtt{activity}]\!]$).

Order-Sorted Feature ($\mathcal{OSF}$) Logic is another formalism also using the notion of conceptual taxonomy as partially-ordered sets [2, 5, 3, 4]. It can also represent attributive information using the notion of *features*, which are functions between concepts. A feature $f : d \rightarrow r$ of a concept d (f's *domain*) maps elements of the set denoted by d to elements of the set denoting concept r (f's *range*). For example, the feature $\mathtt{favoriteHobby} : \mathtt{person} \rightarrow \mathtt{activity}$ can denote *the* activity that is the favorite hobby of a person among all his/her hobbies. Since it denotes a function, a feature $f : a \rightarrow b$ can be also viewed as a role $f : a \times b$ since it also denotes a binary relation between the two set denotations of concepts a and b.

Feature functionality is a crucial property of $\mathcal{OSF}$ Logic because its inference rules, which implement sorted-graph unification (see Section 4, Rule "**FEATURE FUNCTIONALITY**"), rely on it to be correct [2]. This explains our interest in expressing a role as a set-valued feature in $\mathcal{OSF}$ Logic, since this correctly expresses the semantics of roles as binary relations. More specifically, given sets S and S', a binary relation $r \subseteq S \times S'$ can be seen as a function $f_r : S \rightarrow \mathcal{P}(S')$, where $\mathcal{P}(S')$ is the powerset of S'. Then,

$$\forall x \in S,\ f_r(x) \stackrel{\text{DEF}}{=} \{y \in S' \mid \langle x, y\rangle \in r\}.$$

In order to make this operational, we extend $\mathcal{OSF}$ Logic with a new syntactic construct denoting the powerset of a concept. Given a concept c, we write **set-of**(c) the concept denoting the sets of subsets of $[\![c]\!]$; *i.e.*,

$$[\![\textbf{set-of}(c)]\!] \stackrel{\text{DEF}}{=} \mathcal{P}([\![c]\!]).$$

Considering, for example, the `hasHobby` role with domain `person` and range `activity`, this can now be expressed in $\mathcal{OSF}$ Logic as the functional feature:

$$f_{\mathtt{hasHobby}} : \mathtt{person} \rightarrow \textbf{set-of}(\mathtt{activity})$$

to denote the function:

$$[\![f_{\texttt{hasHobby}}]\!] : [\![\texttt{person}]\!] \to \mathcal{P}([\![\texttt{activity}]\!])$$

that associates to a person the set of his/her hobbies if s/he has any.

The purpose of this document is to elaborate this formal semantics given to the **set-of** sort construct, and justify how to use it operationally in the $\mathcal{OSF}$ normalization rules.

The rest of this paper is organized as follows. Section 2 gives a few basic formal definitions needed for the formal semantics of "**set-of**" as powerset used on partialy-ordered sorts. Section 3 shows how to build a "set-complete" domain of partially-ordered sorts from a finite set of atomic partially-ordered sorts. Section 4 is a brief remark on the operational semantics of $\mathcal{OSF}$ normalization with **set-of** sorts. Section 5 discusses relation to other work. We conclude in Section 6.

2 Formal semantics of "set-of"

Let $\langle \mathcal{S}, \leq_{\mathcal{S}}, \top_{\mathcal{S}}, \bot_{\mathcal{S}} \rangle$ be an ordered sort signature, a mathematical structure defined as a finite set of sorts $\mathcal{S}$, partially ordered with an order relation $\leq_{\mathcal{S}}$, and possessing a greatest sort or top element ($\top_{\mathcal{S}}$) and a least sort or bottom element $\bot_{\mathcal{S}}$.

We will use such a sort signature to formalize a concept taxonomy with a most general (*i.e.*, indefinite) concept (denoted by $\top_{\mathcal{S}}$) and an over-defined (*i.e.*, inconsistent) concept (denoted by $\bot_{\mathcal{S}}$), where $s \leq s'$ indicates that the concept denoted by sort s is more defined than, or is subsumed by, the concept denoted by sort s'. In other words, a concept s denotes the set $[\![s]\!]$ of all instances of sort s, and so $s \leq s'$ iff $[\![s]\!] \subseteq [\![s']\!]$.

Let $\mathfrak{A} \stackrel{\text{DEF}}{=} \langle \mathcal{D}^{\mathfrak{A}}, [\![\_]\!]^{\mathfrak{A}} \rangle$ be an interpretation structure defined as a set $\mathcal{D}^{\mathfrak{A}}$ ($\mathfrak{A}$'s domain of interpretation: the *universe of discourse*), and $[\![\_]\!]^{\mathfrak{A}} : \mathcal{S} \to \mathcal{P}(\mathcal{D}^{\mathfrak{A}})$, an interpretation function defining the formal denotational semantics of a sort s in $\mathcal{S}$ as a subset of the domain of interpretation $\mathcal{D}^{\mathfrak{A}}$ such that $[\![\top_{\mathcal{S}}]\!]^{\mathfrak{A}} \stackrel{\text{DEF}}{=} \mathcal{D}^{\mathfrak{A}}$, $[\![\bot_{\mathcal{S}}]\!]^{\mathfrak{A}} \stackrel{\text{DEF}}{=} \varnothing$, and $[\![\leq_{\mathcal{S}}]\!]^{\mathfrak{A}} \stackrel{\text{DEF}}{=} \subseteq$. In other words, for s and s' in $\mathcal{S}$, $s \leq_{\mathcal{S}} s'$ iff $[\![s]\!]^{\mathfrak{A}} \subseteq [\![s']\!]^{\mathfrak{A}}$. If the ordered sort signature is also a lower semilattice (*i.e.*, it has an infimum operation $\wedge_{\mathcal{S}}$), this operation will denote intersection of denotations; *i.e.*, $[\![\wedge_{\mathcal{S}}]\!] \stackrel{\text{DEF}}{=} \cap$. In other words, for s and s' in $\mathcal{S}$, $[\![s \wedge_{\mathcal{S}} s']\!]^{\mathfrak{A}} \stackrel{\text{DEF}}{=} [\![s]\!]^{\mathfrak{A}} \cap [\![s']\!]^{\mathfrak{A}}$. In what follows, to lighten notation, we omit sub/super/scripts such as $\_{\mathcal{S}}$ or $\_^{\mathfrak{A}}$ when obvious from the context.

With the foregoing definitions, the formal semantics of the **set-of** construct on a sort s of a partially-ordered set of sorts is given by:

$$[\![\textbf{set-of}(s)]\!] \stackrel{\text{DEF}}{=} \mathcal{P}([\![s]\!]). \tag{1}$$

In other words, an instance of sort **set-of**(s) denotes a set of instances of sort s.

From this definition, it follows that the order-theoretic properties of sorts extend formally to sets of sorts. Indeed:

PROPOSITION 1 *For all sorts $s \in \mathcal{S}$ and $s' \in \mathcal{S}$, **set-of**$(s) \leq$ **set-of**(s') iff $s \leq s'$.*

PROOF Since the sort subsumption ordering $\leq$ is semantically defined as:

$$s \leq s' \text{ iff } [\![s]\!] \subseteq [\![s']\!]$$

it comes that:

$$\textbf{set-of}(s) \leq \textbf{set-of}(s') \text{ iff } [\![\textbf{set-of}(s)]\!] \subseteq [\![\textbf{set-of}(s')]\!].$$

That is,

$$\textbf{set-of}(s) \leq \textbf{set-of}(s') \text{ iff } \mathcal{P}([\![s]\!]) \subseteq \mathcal{P}([\![s']\!]).$$

From the definition of set intersection in Naive Set Theory [8], it is a simple exercise to show that, for all sets A and B, $A \subseteq B$ iff $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Therefore,

$$\textbf{set-of}(s) \leq \textbf{set-of}(s') \text{ iff } [\![s]\!] \subseteq [\![s']\!],$$

and so:

$$\textbf{set-of}(s) \leq \textbf{set-of}(s') \text{ iff } s \leq s'. \qquad \square$$

COROLLARY 1 *For all sorts $s \in \mathcal{S}$, **set-of**$(\bot_{\mathcal{S}}) \leq$ **set-of**$(s) \leq$ **set-of**$(\top_{\mathcal{S}})$.*

In other words, the ordered set **set-of**$(\mathcal{S})$ has a greatest element:

$$\top_{\textbf{set-of}(\mathcal{S})} \stackrel{\text{DEF}}{=} \textbf{set-of}(\top_{\mathcal{S}})$$

and a least element:

$$\bot_{\textbf{set-of}(\mathcal{S})} \stackrel{\text{DEF}}{=} \textbf{set-of}(\bot_{\mathcal{S}}).$$

If the ordered sort structure $\mathcal{S}$ is also a lower semi-lattice structure for $\leq_{\mathcal{S}}$ with infimum operation $\wedge_{\mathcal{S}}$ such that $[\![\wedge_{\mathcal{S}}]\!]^{\mathfrak{A}} \stackrel{\text{DEF}}{=} \cap$ in $\mathcal{P}(\mathcal{D}^{\mathfrak{A}})$ (for an appropriate interpretation $\mathfrak{A}$), then:

PROPOSITION 2 *For all sorts $s \in \mathcal{S}$ and $s' \in \mathcal{S}$,*

$$[\![\textbf{\textit{set-of}}(s \wedge s')]\!] = [\![\textbf{\textit{set-of}}(s) \wedge \textbf{\textit{set-of}}(s')]\!].$$

PROOF By definition:

$$[\![\textbf{set-of}(s \wedge s')]\!] = \mathcal{P}([\![s \wedge s']\!]);$$

and so, again by definition:

$$[\![\textbf{set-of}(s \wedge s')]\!] = \mathcal{P}([\![s]\!] \cap [\![s']\!]).$$

From the definition of the powerset of a set in Naive Set Theory fact [8], it is not a difficult exercise to show that if A and B are two sets then,

$$\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B). \qquad (2)$$

Therefore:

$$[\![\textbf{set-of}(s \wedge s')]\!] = \mathcal{P}([\![s]\!]) \cap \mathcal{P}([\![s']\!]). \tag{3}$$

That is:

$$[\![\textbf{set-of}(s \wedge s')]\!] = [\![\textbf{set-of}(s)]\!] \cap [\![\textbf{set-of}(s')]\!].$$

And hence:

$$[\![\textbf{set-of}(s \wedge s')]\!] = [\![\textbf{set-of}(s) \wedge \textbf{set-of}(s')]\!]. \qquad \square$$

Note that the equation obtained by using set union instead of set intersection in Equation (2) does not hold. This is because the powerset of the union of two sets will contain all the subsets of each, but also the sets that are subsets of neither while having elements from both. However, the following holds: $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. If neither $A \neq \varnothing$ nor $B \neq \varnothing$, this containment is strict; that is, if $A \neq \varnothing$ and $B \neq \varnothing$, then $\mathcal{P}(A) \cup \mathcal{P}(B) \subset \mathcal{P}(A \cup B)$. In other words, for any set S, there is a lower-semilattice homomorphism between $\langle \mathcal{P}(S), \subseteq \rangle$ and $\langle \mathcal{P}(\mathcal{P}(S)), \subseteq \rangle$, but it is not a lattice homomorphism.

Because the denotational semantic function $[\![\_]\!] : \mathcal{S} \rightarrow \mathcal{D}_{\mathfrak{A}}$ engenders a syntactic congruence on sorts $\simeq_{\mathcal{S}}$ defined as:

$$s \simeq_{\mathcal{S}} s' \;\text{ iff }\; [\![s]\!] = [\![s']\!], \tag{4}$$

the following syntactic sort congruence is semantically justified by Proposition 2:

COROLLARY 2 *For all sorts $s \in \mathcal{S}$ and $s' \in \mathcal{S}$,*

$$\textbf{\textit{set-of}}(s) \wedge \textbf{\textit{set-of}}(s') \;\simeq_{\mathcal{S}}\; \textbf{\textit{set-of}}(s \wedge s').$$

3 Set-complete domains

3.1 Informal motivation

The formal semantics of the **set-of** constructor on a set of partially ordered sorts presented in Section 2 fits our (naive) intuition. However, there are legitimate questions regarding whether there exists a semantic model comprising sets that are denotable by a syntactic expression of the form $\textbf{set-of}^{\,n}(s)$ for some sort s (where $n \in \mathbb{N}$, and $\textbf{set-of}^{\,0}(s) \stackrel{\text{DEF}}{=} s$), and if so, whether one can be built from a model of atomic sorts. We shall call such a model, a *set-complete* model.

However, since we rely on set-theoretic arguments to justify formal aspects of this semantics, it is important to be very precise about *which* Set Theory we refer to. Indeed, there are several mathematical set theories, depending on what axioms they admit. This is important since each specific axiom will determine fundamental properties of the Set Theory in which it holds.

Now, we do not intend (nor would we be competent) to give here a full formal account and analysis of all potential varieties of set theories. This has occupied a great number of the greatest mathematicians and philosophers over the past 140 years—since the ideas of Georg Cantor were originally published [9]. We are however interested in clarifying the nature of the sets denoted by our specific **set-of** syntactic constructs depending on that of the Set Theory considered.

In the previous section, we have casually invoked Naive Set Theory to justify our semantics of **set-of**. This set theory may be construed as our intuitive understanding of the notion of set as a collection of elements. But one must be careful. Otherwise, as was pointed out by Bertrand Russell in [13], this leads to an inconsistent set theory—caused by the so-called "Set-of-All-Sets Paradox." This paradox states that if a theory of sets admits that a set can contain itself, the theory becomes inconsistent.

Quoting Bertrand Russell [14]:

> *"There is a greatest cardinal in each type, namely the cardinal number of the whole of the type; but this is always surpassed by the cardinal number of the next type, since, if α is the cardinal number of one type, that of the next type is 2^α, which, as Cantor has shown, is always greater than α."*

But if $\mathcal{P}(S) \subseteq S$, for some S, this entails that $|\mathcal{P}(S)| \leq |S|$; that is, $2^{|S|} \leq |S|$ (where $|S|$ denotes the cardinal of S). And so by Cantor as quoted by Russell, a contradiction follows.

To circumvent this problem, *Axiomatic Set Theory*—specifically, Zermelo-Fraenkel (ZF) Set Theory [10]—was introduced to specify formally exactly what properties can, or not, be held valid in a set theory. Hence, to prevent Russell's paradox of Naive Set Theory, ZF admits a specific axiom: the *axiom of well-foundedness* (also called *axiom of regularity*), which states that, for any set x:

$$x \notin x. \tag{5}$$

Thus, any axiomatic set theory that admits this axiom is deemed a *well-founded* set theory, since it prevents Russell's paradox. However, there are also axiomatic set theories that are non well-founded set theories. These set theories do not admit Axiom (5). In such theories, it is admissible for a set to be a member of itself, yet to be consistent. An early example is Quine's New Foundations [12]. A more recent example is Peter Aczel's [1]. In these set theories, for example, the one-element set $x = \{x\}$ is a perfectly valid object—one which could be used in Computer Science to denote a circular object pointing to itself, for example.

So, coming back to our concerns regarding the semantics of **set-of**, we must then first clarify the kinds of sets that we wish to represent before worrying about which Set Theory to use for their semantics.

We assume a universe of discourse populated by distinguishable individuals we shall call "atomic instances." Examples of such are the natural numbers; animals; flowers; people; countries; and so forth. These individuals will be deemed of *type order* 0 to indicate that each instance is atomic; *i.e.*, it cannot be decomposed into further components (at least as far as we are interested). Then, we can consider sets of such atomic instances; *e.g.*, sets of natural numbers; sets of animals; sets of countries; *etc.*, … Such sets of atomic instances will be deemed of type order 1 to indicate that their elements are instances of type order 0. Note that these describe sets of elements of equal type orders. This is not the case of the set $\{1, \{1\}\}$, for example; or, sets composed both of students and of sets of students. We shall express this property of a semantic domain as being of *uniform* type order. We can of course iterate the process of building sets of higher order types containing such uniformly typed elements.

From these considerations, it appears that what constitutes a "set-complete" domain extending an interpretation domain of atomic individual instances should be carefully analyzed, and its construction, if at all possible, formally justified. We proceed to do so next.

3.2 A formal set-complete domain construction

We will start by analyzing formally the consequences of our informal foregoing requirements for a domain to be "set-complete."

Let $\langle \mathcal{S}, \leq_{\mathcal{S}}, \top_{\mathcal{S}}, \bot_{\mathcal{S}} \rangle$ be an ordered sort signature, and $\mathfrak{A} \overset{\text{DEF}}{=} \langle \mathcal{D}^{\mathfrak{A}}, [\![\_]\!]^{\mathfrak{A}} \rangle$ be an interpretation structure, with denotation function $[\![\_]\!] : \mathcal{S} \rightarrow \mathcal{P}(\mathcal{D}^{\mathfrak{A}})$. If we define the denotation of $\top_{\mathcal{S}}$ as $[\![\top_{\mathcal{S}}]\!]^{\mathfrak{A}} \overset{\text{DEF}}{=} \mathcal{D}^{\mathfrak{A}}$ and if we define the set $\mathcal{S}$ to be closed under the **set-of** constructor defined in Section 2, this would mean in particular that:

$$\textbf{set-of}\,(\top_{\mathcal{S}}) \leq \top_{\mathcal{S}}. \tag{6}$$

But, by the semantics given by Equation (1), this would mean:

$$\mathcal{P}(\mathcal{D}^{\mathfrak{A}}) \subseteq \mathcal{D}^{\mathfrak{A}} \tag{7}$$

or, equivalently:

$$\mathcal{P}(\mathcal{D}^{\mathfrak{A}}) \in \mathcal{P}(\mathcal{D}^{\mathfrak{A}}). \tag{8}$$

And this, as Russell argued, leads to a paradox. We next proceed to show that we need not be concerned by such a paradox. This is because we are not interested in arbitrary subsets, but only in *subsets of uniform type orders*. So let us proceed carefully in order to define an order structure for the sets of all sorts $\mathcal{S}$ to meet our requirements, starting with atomic sorts.

Let:

$$\langle \mathcal{A}, \leq_{\mathcal{A}}, \top_{\mathcal{A}}, \bot_{\mathcal{A}} \rangle \tag{9}$$

be a partially ordered signature denoting *atomic* concepts. $\mathcal{A}$ is a finite set of *atomic* sort symbols in $\mathcal{S}$, including $\top_{\mathcal{A}}$ the greatest atomic sort, and $\bot_{\mathcal{A}}$ the least atomic sort. Given an atomic sort in $\mathcal{A}$ (say, `integer`), the notation **set-of** (`integer`) will denote the sort of sets of elements of sort `integer`. The constructor **set-of** may also be used on the sort of a set; *e.g.*, **set-of** (**set-of** (`integer`)) will denote the set of sets of elements of sort `integer`. Thus, the set of all sorts $\mathcal{S}$ consists of the set $\mathcal{A}$, as well as the higher-order powersort structures **set-of**$(\mathcal{A})$, **set-of**(**set-of**$(\mathcal{A})$), *etc.*, ...

But because we wish the set of all sorts $\mathcal{S}$ also to have a greatest sort and a least sort, we add to it two new sort symbols, written $\top$ and $\bot$, such that, for all $n \in \mathbb{N}$:

$$\top_{\textbf{set-of}^{\,n}(\mathcal{A})} \leq \top \tag{10}$$

and

$$\bot \leq \bot_{\textbf{set-of}^{\,n}(\mathcal{A})}. \tag{11}$$

Again, we use the convention that, for $n = 0$, **set-of**$^{\,0}$ is the identity function.

Therefore, if we define $\mathcal{S}_n \stackrel{\text{DEF}}{=} \textbf{set-of}^{\,n}(\mathcal{A})$, this requirement is that the set of sorts $\mathcal{S}$ must be such that:

$$\{\top, \bot\} \cup \bigcup_{n\in\mathbb{N}} \textbf{set-of}^{\,n}(\mathcal{A}) \subseteq \mathcal{S}. \tag{12}$$

But this is not sufficient to meet our requirement that the set of all sorts $\mathcal{S}$ must also contain **set-of**(s) for any $s \in \mathcal{S}$. This is because this requirement entails inductively that we must also have:

$$\{\textbf{set-of}^{\,n}(\top), \textbf{set-of}^{\,n}(\bot)\} \cup \bigcup_{n\in\mathbb{N}} \mathcal{S}_n \subseteq \mathcal{S}. \tag{13}$$

Therefore, this defines the whole set of sorts $\mathcal{S}$ as:

$$\mathcal{S} \stackrel{\text{DEF}}{=} \bigcup_{n\in\mathbb{N}} \textbf{set-of}^{\,n}(\mathcal{A}) \cup \{\textbf{set-of}^{\,n}(\top) \mid n \in \mathbb{N}\} \cup \{\textbf{set-of}^{\,n}(\bot) \mid n \in \mathbb{N}\} \tag{14}$$

As explained, the objective of our construction is to make a distinction between uniform and non-uniform type orders of sets, keeping $\mathcal{S}$ strictly stratified according to the *type order* of its sorts. What follows elaborates the formal details of what this entails.

3.2.1 Greatest sorts of uniform type order

Let us first pay attention to the top elements.

PROPOSITION 3 *For all* $n \in \mathbb{N}$,

$$\top_{\textbf{\textit{set-of}}^{\,n}(\mathcal{A})} = \textbf{\textit{set-of}}^{\,n}(\top_{\mathcal{A}}).$$

PROOF Proceed by induction on n. □

For this reason, we use the notation $\top_n \stackrel{\text{DEF}}{=} \textbf{set-of}^{\,n}(\top_{\mathcal{A}})$.

Hence, using Corollary 1 inductively on successive powers of **set-of** justifies defining the top element of each **set-of** power as the power of the top element of the domain of immediately lower type order.

DEFINITION 1 *For all* $n \in \mathbb{N}$*:*

$$\top_n \stackrel{\text{DEF}}{=} \top_{\textbf{\textit{set-of}}^{\,n}(\mathcal{A})} = \begin{cases} \top_{\mathcal{A}} & \textit{if } n = 0; \\ \textbf{\textit{set-of}}(\top_{n-1}) & \textit{if } n > 0. \end{cases}$$

Following what we defined above and in Section 2, we have,

PROPOSITION 4 *For all* $n \in \mathbb{N}$,

$$[\![\top_n]\!] = \mathcal{P}^n([\![\mathcal{A}]\!]).$$

PROOF By definition, $\top_0 \stackrel{\text{DEF}}{=} \top_{\mathcal{A}}$, and $\top_n \stackrel{\text{DEF}}{=} \textbf{set-of}^{\,n}(\top_0)$. Use Equation (1) and proceed by induction on n. □

In other words, the symbol $\top_0$ denotes the set of all atomic elements (*i.e.*, $[\![\top_0]\!] = [\![\textbf{set-of}^{\,0}(\mathcal{A})]\!] = [\![\mathcal{A}]\!]$), and the symbol $\top_n$ denotes the set of all sets of elements in $[\![\top_{n-1}]\!]$, for $n > 1$.

Note that, for any $n \in \mathbb{N}$, the sort $\top_n$ denotes a set of elements of uniform type order n; and so do all its subsorts. However, this is not true of their super sorts $\textbf{set-of}^{\,n}(\top)$, which are, however, strictly stratified as explained next.

3.2.2 Sorts of non uniform type order

Consider now the sorts $\textbf{set-of}^{\,n}(\top)$, for $n \in \mathbb{N}$, that we added in the definition of $\mathcal{S}$ given by Equation (14). Their denotation is defined as:

$$[\![\textbf{set-of}^{\,n}(\top)]\!] \stackrel{\text{DEF}}{=} \bigcup_{m \geq n} [\![\top_m]\!] = \bigcup_{m \geq n} [\![\textbf{set-of}^{\,m}(\top_{\mathcal{A}})]\!] = \bigcup_{m \geq n} \mathcal{P}^m([\![\mathcal{A}]\!]). \quad (15)$$

Note that these sorts have elements of different order types; and so they are non-uniformly typed. However, they are strictly stratified in a way to be made clear next.

Setting $n = 0$ in Equation (15), this entails that:

$$[\![\top]\!] = \bigcup_{n \in \mathbb{N}} [\![\top_n]\!] = \bigcup_{n \in \mathbb{N}} \mathcal{P}^n([\![\mathcal{A}]\!]). \quad (16)$$

From Equation (15), it also follows that, for all n and m in $\mathbb{N}$:

$$n < m \implies [\![\textbf{set-of}^{\,m}(\top)]\!] \subset [\![\textbf{set-of}^{\,n}(\top)]\!]. \quad (17)$$

In other words, the sort denotations $[\![\textbf{set-of}^{\,n}(\top)]\!]$, for $n \in \mathbb{N}$, form a strictly *descending* infinite chain for the $\subset$ ordering:

$$[\![\top]\!] \supset [\![\textbf{set-of}(\top)]\!] \supset \ldots [\![\textbf{set-of}^{\,2}(\top)]\!] \supset \ldots [\![\textbf{set-of}^{\,n}(\top)]\!] \supset \ldots$$

Hence, this justifies semantically the same fact for the syntactic sorts $\textbf{set-of}^{\,n}(\top)$ with the $<_{\mathcal{S}}$ sort ordering. Namely,

$$\top > \textbf{set-of}(\top) > \ldots \textbf{set-of}^{\,2}(\top) > \ldots \textbf{set-of}^{\,n}(\top) > \ldots$$

In particular, for all $n \geq 1$:

$$[\![\textbf{set-of}^{\,n}(\top)]\!] \subset [\![\top]\!] \tag{18}$$

and

$$\textbf{set-of}^{\,n}(\top) < \top. \tag{19}$$

Note that, for $n = 1$, Inequality (18) is the same as Inequality (6). However, it does not give rise to a paradox since the semantics of $\top$ given by Equation (16) is not the powerset of the whole domain but the disjoint union of stratified powers of different type orders of the domain of atomic sorts. Indeed, in our construction, Equation (1) holds only for a sort $s \in \bigcup_{n\in\mathbb{N}} \textbf{set-of}^{\,n}(\mathcal{A})$. It does not hold for a sort $s \in \{\textbf{set-of}^{\,n}(\top) \mid n \in \mathbb{N}\}$, for which the semantics is given by Equation (15).

3.2.3 Least sorts of uniform type order

Let us now look at the bottom elements.

PROPOSITION 5 *For all* $n \in \mathbb{N}$, $\perp_{\textbf{\textit{set-of}}^{\,n}(\mathcal{A})} = \textbf{\textit{set-of}}^{\,n}(\perp_{\mathcal{A}})$.

PROOF Proceed by induction on n. □

For this reason, we use the notation $\perp_n \stackrel{\text{DEF}}{=} \textbf{set-of}^{\,n}(\perp_{\mathcal{A}})$.

Similarly to what we did for the top elements, we define a lowest **set-of** sorts of type order n, $\perp_n \in \mathcal{S}_n$, $(n \in \mathbb{N})$; *viz.*, the bottom element of each **set-of** power is the power of the bottom element of the domain of immediately lower type order.

DEFINITION 2 *For all* $n \in \mathbb{N}$*:*

$$\perp_n \stackrel{\text{DEF}}{=} \perp_{\textbf{\textit{set-of}}^{\,n}(\mathcal{A})} = \begin{cases} \perp_{\mathcal{A}} & \textit{if } n = 0; \\ \textbf{\textit{set-of}}(\perp_{n-1}) & \textit{if } n > 0. \end{cases}$$

Therefore, the semantics of the $\perp_n$ sorts $(n \in \mathbb{N})$ is defined as follows.

PROPOSITION 6 *For all* $n \in \mathbb{N}$, $[\![\perp_n]\!] = \mathcal{P}^n(\varnothing)$.

PROOF By definition, $\perp_0 \stackrel{\text{DEF}}{=} \perp_{\mathcal{A}}$ and $\perp_n \stackrel{\text{DEF}}{=} \textbf{set-of}^{\,n}(\perp_0)$. Then, proceed by induction using Equation (1). □

The symbol $\perp_0 \stackrel{\text{DEF}}{=} \perp_{\mathcal{A}}$ denotes the empty set (by definition of the lowest element of $\mathcal{A}$); so we have $[\![\perp_0]\!] = \varnothing$. Importantly, this means that there is no need to add a *new* symbol $\perp$ to the set $\mathcal{S}$ as specified in Definition 14. Indeed, the need for $\perp$ was to have a least sort. Thus, this least sort must therefore be such $\perp \leq \perp_0$. Semantically, this means that $[\![\perp]\!] \subseteq [\![\perp_0]\!]$. But $[\![\perp_0]\!] \stackrel{\text{DEF}}{=} \varnothing$. Therefore, by equation (4), this entails necessarily that $\perp \simeq_{\mathcal{S}} \perp_0$.

Now, by Proposition 6, $[\![\perp_1]\!] = \mathcal{P}([\![\perp_0]\!]) = \mathcal{P}(\varnothing)$. But,

$$\mathcal{P}(\varnothing) = \{\varnothing\} \neq \varnothing. \tag{20}$$

In other words, the powerset of the empty set is the singleton set containing the empty set, and it is therefore non empty.

From Proposition 6, it also follows that, for all m and n in $\mathbb{N}$:

$$m < n \implies [\![\textbf{set-of}^{\,m}(\perp)]\!] \subset [\![\textbf{set-of}^{\,n}(\perp)]\!]. \tag{21}$$

In other words, the sets $[\![\perp_n]\!]$, for $n \in \mathbb{N}$ form a strictly *ascending* chain for the $\subset$ ordering. Namely:

PROPOSITION 7 *For all* $n \in \mathbb{N}$, $[\![\perp_n]\!] \subset [\![\perp_{n+1}]\!]$.

> PROOF Proceed by induction on n. This is clearly true for $n = 0$. Assume now that $[\![\perp_n]\!] \subset [\![\perp_{n+1}]\!]$. Then, from Proposition 1, this entails that $\mathcal{P}([\![\perp_n]\!]) \subset \mathcal{P}([\![\perp_{n+1}]\!])$; that is, $[\![\perp_{n+1}]\!] \subset [\![\perp_{n+2}]\!]$. □

This means that:

$$[\![\perp_0]\!] \subset [\![\perp_1]\!] \subset \ldots [\![\perp_n]\!] \subset \ldots$$

which semantically justifies the syntactic sort ordering:

$$\perp_0 < \perp_1 < \ldots \perp_n < \ldots$$

COROLLARY 3 *For all* $n \in \mathbb{N}$, $\perp_n < \perp_{n+1}$.

Figure 1 depicts the subsort order structure of such a set-complete sort domain obtained by the above construction from a set of partially-ordered atomic sorts.

We are now in a position to define the set-complete semantic interpretation domain $[\![\mathcal{S}]\!]$ as the set:

$$[\![\mathcal{S}]\!] \stackrel{\text{DEF}}{=} \bigcup_{n\in\mathbb{N}} \mathcal{P}^{\,n}([\![\mathcal{A}]\!]) \,\cup\, \left\{ \bigcup_{m\geq n} \mathcal{P}^{\,m}([\![\mathcal{A}]\!]) \mid n \in \mathbb{N} \right\} \tag{22}$$

with, for all $n \in \mathbb{N}$:

- $\forall a \in \mathcal{A}$, $[\![\textbf{set-of}^{\,n}(a)]\!] \stackrel{\text{DEF}}{=} \mathcal{P}^{\,n}([\![a]\!])$;
- $[\![\textbf{set-of}^{\,n}(\top)]\!] \stackrel{\text{DEF}}{=} \bigcup_{m\geq n} \mathcal{P}^{\,m}([\![\mathcal{A}]\!])$;
- $[\![\textbf{set-of}^{\,n}(\perp)]\!] \stackrel{\text{DEF}}{=} \mathcal{P}^{\,n}(\varnothing)$.

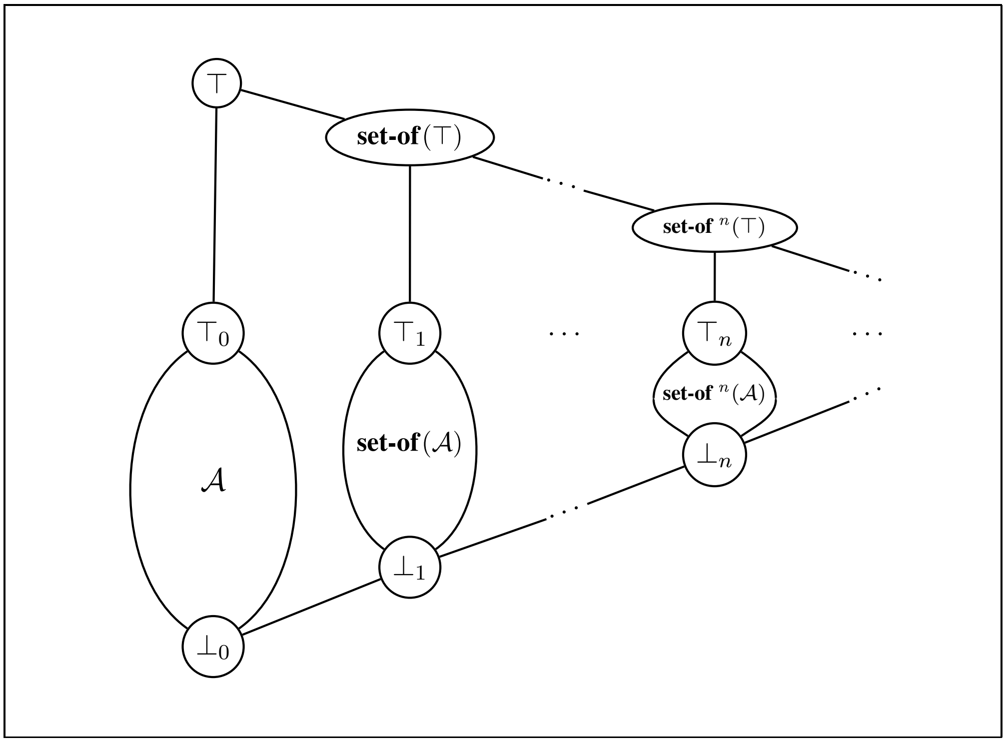

Figure 1: The order structure of a set-complete sort signature

4 Operational semantics

As for operational semantics, the basic $\mathcal{OSF}$ constraint normalization rules, given in [2] and recalled in Figure 2, can be used with a set-complete sort signature $\mathcal{S}$ since the rule "**SORT INTERSECTION**" will work correctly as semantically justified by Corollary 2.

(1) **SORT INTERSECTION**:

$$\frac{\phi \;\&\; X : s \;\&\; X : s'}{X : s \wedge s'}$$

(2) **FEATURE FUNCTIONALITY**:

$$\frac{\phi \;\&\; X.f \doteq X' \;\&\; X.f \doteq X''}{\phi \;\&\; X.f \doteq X' \;\&\; X' \doteq X''}$$

(3) **INCONSISTENT SORT**:

$$\frac{\phi \;\&\; X : \perp}{X : \perp}$$

(4) **VARIABLE ELIMINATION**:

$$\frac{\phi \;\&\; X \doteq X'}{\phi[X'/X] \;\&\; X \doteq X'} \;\text{[if } X' \text{ occurs in } \phi\text{]}$$

Figure 2: Basic $\mathcal{OSF}$-constraint normalization rules

5 Relation to Other Work

There is classical work in Domain Theory giving formal constructions of powerdomains for Denotational Semantics (see for example [11, 16]). However, the concerns of these constructions are in the context of denotational approximation orderings, not set inclusion as is the case in this work.

As explained in [7] (pages 23–29), there are three kinds of powerdomains that can be defined in Denotational Semantics for a domain $\langle \mathcal{D}, \sqsubseteq \rangle$:

- the *lower* powerdomain $\mathcal{D}^\flat \stackrel{\text{DEF}}{=} \langle \mathcal{P}(\mathcal{D}), \sqsubseteq_\flat \rangle$, where:

  $$u \sqsubseteq_\flat v \text{ iff } \forall x \in u, \exists y \in v, \text{ s.t. } x \sqsubseteq y$$

  (this is also known as the *Hoare powerdomain* of $\mathcal{D}$);

- the *upper* powerdomain $\mathcal{D}^\sharp \stackrel{\text{DEF}}{=} \langle \mathcal{P}(\mathcal{D}), \sqsubseteq_\sharp \rangle$, where:

  $$u \sqsubseteq_\sharp v \text{ iff } \forall y \in v, \exists x \in u, \text{ s.t. } x \sqsubseteq y$$

  (this is also known as the *Smyth powerdomain* of $\mathcal{D}$);

- the *convex* powerdomain $\mathcal{D}^\natural \stackrel{\text{DEF}}{=} \langle \mathcal{P}(\mathcal{D}), \sqsubseteq_\natural \rangle$, where:

  $$u \sqsubseteq_\natural v \text{ iff } u \sqsubseteq_\flat v \text{ and } u \sqsubseteq_\sharp v$$

  (this is also known as the *Plotkin powerdomain* of $\mathcal{D}$);

for u and v in $\mathcal{P}(\mathcal{D})$. The sets in these powerdomains denote non-deterministic choices of elements of the underlying domain: each of these three constructions captures a different kind of non-determinism.

The concern of these constructions is not to extend powerdomains to arbitrary higher types. Rather, it aims at establishing that these constructions preserve continuity of the approximation ordering and existence of limits on non-deterministic sets.

Whereas, our construction purports to define a "powersort" **set-of**(s), for a given sort s, as the set of subsets of the set of elements denoted by s. These are not non-deterministic choice sets ordered with a continuous approximation ordering as in denotational semantics, but plain deterministic sets ordered by set inclusion—just as the underlying sorts are. As such, the ordering on "**set-of**" sorts is a straightforward extension of the ordering on the underlying sorts; namely, **set-of**$(s) \leq$ **set-of**(s') iff $s \leq s'$. However, unlike the denotational semantics powerdomain constructions cited above, our concern is with closing our construction to sets of arbitrary order types. In so doing, we wished to ensure that consistent (*i.e.*, paradox-free) interpretation domains actually exist and can be built—which we established using a stratification technique.

6 Conclusion

We have presented a formal construction allowing "powersorts" in a lower semilattice of sorts using a "**set-of**" constructor on sorts. We showed that such a construct on atomic set-denoting sorts naturally extends their ordering denoting set inclusion. The main result presented in this paper is that this construction gives well-defined denotations for sets of arbitrary order types. The motivation for this work has been to use set-valued functions for representing binary relation properties known as roles in Semantic Web formalisms based on Description Logic. This construction is novel as far as we know. Classical work in Domain Theory such as Plotkin's [11] and Scott's [16] are, although concerned with similar issues for extending a semantic domain's approximation ordering to non-deterministic sets, not concerned with the same issues. Even though our concern is much more modest in scope and intent than the cited constructions of Denotational Semantics, this contribution is necessary as a formal justification for using set-valued features in the $\mathcal{OSF}$ formalism in order to extend it to support relational attributes of partially ordered sorts of the kind used in Description Logic.

References

[1] Peter Aczel. *Non Well-Founded Sets*. Center for the Study of Language and Information, Stanford, CA, USA, 1988. [See online[1]].

[2] Hassan Aït-Kaci. Data models as constraint systems: A key to the Semantic Web. *Constraint Processing Letters*, 1:33–88, November 2007. [See online[2]].

[3] Hassan Aït-Kaci and Andreas Podelski. Functions as passive constraints in LIFE. *ACM Transactions on Programming Languages and Systems*, 16(4):1279–1318, July 1994. [See online[3]].

[4] Hassan Aït-Kaci, Andreas Podelski, and Seth C. Goldstein. Order-sorted feature theory unification. *Journal of Logic Programming*, 30(2):99–124, 1997. [See online[4]].

[5] Hassan Aït-Kaci, Andreas Podelski, and Gert Smolka. A feature-based constraint system for logic programming with entailment. *Theoretical Computer Science*, 122(1–2):263–283, January 1994. [See online[5]].

[6] Sean Bechhofer, Frank van Harmelen, James Hendler, Ian Horrocks, Deborah L. McGuinness, Peter F. Patel-Schneider, and Lynn Andreas Stein. OWL web ontology language reference. W3C Recommendation. Mike Dean and Guus Schreiber, eds., February 2004. [See online[6]].

[7] Carl A. Gunter, Peter D. Mosses, and Dana S. Scott. Semantic domains and denotational semantics. Technical Report MS-CIS-89-16, Department of Computer and Information Science, University of Pennsylvania, Philadelphia, PA, USA, February 1989. [See online[7]].

[1] http://standish.stanford.edu/pdf/00000056.pdf
[2] http://cs.brown.edu/people/pvh/CPL/Papers/v1/hak.pdf
[3] http://hassan-ait-kaci.net/pdf/toplas-94.pdf
[4] http://hassan-ait-kaci.net/pdf/osf-theory-unification.pdf
[5] http://hassan-ait-kaci.net/pdf/tcs-94.pdf
[6] http://www.w3.org/TR/owl-ref/
[7] http://repository.upenn.edu/cgi/view...?article=1887&context=cis_reports

[8] Paul R. Halmos. *Naive Set Theory*. Springer, 1974. [See online[8]].

[9] Phillip E. Johnson. The genesis and development of set theory. *The Two-Year College Mathematics Journal*, 3(1):55*ff*, Spring 1972. [See online[9]].

[10] Tony Lian. Fundamentals of Zermelo-Fraenkel set theory. Apprentice Program Lecture Notes, Research Experiences for Undergraduates, National Science Foundation's Vertical Integration of Research and Education in the Mathematical Sciences, Department of Mathematics, The Universiy of Chicago, May 2011. [See online[10]].

[11] Gordon D. Plotkin. A powerdomain construction. *SIAM Journal on Computing*, 5(3):452–487, September 1976. [See online[11]].

[12] Willard Van Orman Quine. New foundations for mathematical logic. *American Mathematical Monthly*, 44(2):70–80, February 1937.

[13] Bertrand Russell. *The Principles of Mathematics*. Cambridge University Press, 1903. [See online[12]].

[14] Bertrand Russell. *Logic and Knowledge: Essays 1901–1950*. edited by Robert C. Marsh. George Allen & Unwin Publishers, London, UK, 1956. Chapter 3: "*1908: Mathematical Logic as Based on The Theory of Types*," pp. 57–102. [See online[13]].

[15] Manfred Schmidt-Schauß and Gert Smolka. Attributive concept descriptions with complements. *Artificial Intelligence*, 48:1–26, 1991. [See online[14]].

[16] Dana S. Scott. Data types as lattices. *SIAM Journal on Computing*, 5(3):522–587, September 1976. [See online[15]].

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[8] http://www.math.usu.edu/~rheal/math4200/class_material/set_axioms.pdf
[9] http://www.jstor.org/discover/10.2307/3026799?uid=2134&uid=383556181
[10] http://www.math.uchicago.edu/~may/VIGRE/VIGRE2011/REUPapers/Lian.pdf
[11] http://homepages.inf.ed.ac.uk/gdp/publications/Powerdomain_Construction.pdf
[12] http://fair-use.org/bertrand-russell/the-principles-of-mathematics/
[13] http://books.google.fr/books?id=GK6aU0VUgXcC&pg=PA97
[14] http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.397.6454
[15] https://www.cs.ox.ac.uk/files/3287/PRG05.pdf

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