37 Graphs

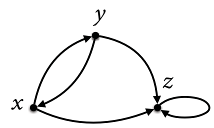
The Roy-Floyd–Warshall algorithm [8, 14, 18] finds shortest paths in a weighted graph (with no cycle with strictly negative weight). It is an abstract interpretation of a fixpoint path finding algorithm. [2, 3, 6, 15] are introductions to graph theory.

Contents

37.1 Graphs

A (directed) *graph* or *digraph* $G = \langle V, E\rangle$ is a pair of a set V of *vertices* (or *nodes* or *points*) and a set $E \in \wp(V \times V)$ of *edges* (or *arcs*). A (directed) edge $\langle x, y\rangle \in V$ has *origin* x and *end* y collectively called *extremities* (so the graphs we consider are always directed). Therefore E is a binary relation of Section **2.2.2** on V. Conversely, a binary relation can be understood as a graph on its field [15]. A graph is *finite* when the set of V of vertices (hence E) is finite.

Example 37.1



$$G = \left[\begin{array}{lcl} V & = & \{x, y, z\} \\ E & = & \{\langle x,\ y\rangle, \langle x,\ z\rangle, \langle y,\ x\rangle, \\ & & \langle y,\ z\rangle, \langle z,\ z\rangle\} \end{array}\right] \qquad \mathbf{G} = \left[\begin{array}{c|ccc} & x & y & z \\ \hline x & 0 & 1 & 1 \\ y & 1 & 0 & 1 \\ z & 0 & 0 & 1 \end{array}\right]$$

□

37.2 Paths and cycles of a graph

A *path* π from y to z in a graph $G = \langle V,\ E\rangle$ is a finite sequence of vertices $\pi = x_1 \dots x_n \in V^n, n > 1$, starting at *origin* $y = x_1$, finishing at *end* $z = x_n$, and linked by edges $\langle x_i,\ x_{i+1}\rangle \in E, i \in [1, n[$[44]. Let $V^{>1} \triangleq \bigcup_{n>1} V^n$ be the sequences of vertices of length at least 2. Formally the set $\Pi(G) \in \wp(V^{>1})$ of all paths of a graph $G = \langle V,\ E\rangle$ is

$$\begin{aligned} \Pi(G) &\triangleq \bigcup_{n>1} \Pi^n(G) && (37.2) \\ \Pi^n(G) &\triangleq \{x_1 \dots x_n \in V^n \mid \forall i \in [1, n[\ .\ \langle x_i,\ x_{i+1}\rangle \in E\} \qquad (n > 1) \end{aligned}$$

The length $|\pi|$ of the path $\pi = x_1 \dots x_n \in V^n$ is the number of edges that is $n - 1 > 0$. We do not consider the case $n = 1$ of paths of length 0 with only one vertex since paths must have at least one edge. A *subpath* is forming a strict part of another path (which, being strict, is not equal to that path).

The *vertices of a path* $\pi = x_1 \dots x_n \in \Pi^n(G)$ of a graph G is the set $\mathsf{V}(\pi) = \{x_1 \dots x_n\}$ of vertices appearing in that path π.

A *cycle* is a path $x_1 \dots x_n \in \Pi^n(G)$ with $x_n = x_1, n > 1$. Self-loops *i.e.* $\langle x,\ x\rangle \in E$ yield a cycle $x\,x$ of length 1.

37.3 Fixpoint characterization of the paths of a graph

The concatenation of sets of finite paths is

$$P \circledcirc Q \quad \triangleq \quad \{x_1 \dots x_n y_2 \dots y_m \mid x_1 \dots x_n \in P \land x_n y_2 \dots y_m \in Q\}. \tag{37.3}$$

We have the following fixpoint characterization of the paths of a graph [4, Sect. 4], which is well-defined, by Tarski iterative fixpoint Theorem 13.18 .

[44] In order to consider infinite paths, we would have to introduce limits as in Section 7.2.

Theorem 37.4 (Fixpoint characterization of the paths of a graph) The paths of a graph $G = \langle V,\ E\rangle$ are

$$
\begin{array}{rcllcll}
\Pi(G) & = & \mathsf{lfp}^{\subseteq}\, \overrightarrow{\mathcal{F}}_{\Pi}, & \overrightarrow{\mathcal{F}}_{\Pi}(X) & \triangleq & E \cup X \circledcirc E & (37.4.a)\\
 & = & \mathsf{lfp}^{\subseteq}\, \overleftarrow{\mathcal{F}}_{\Pi}, & \overleftarrow{\mathcal{F}}_{\Pi}(X) & \triangleq & E \cup E \circledcirc X & (37.4.b)\\
 & = & \mathsf{lfp}^{\subseteq}\, \overleftrightarrow{\mathcal{F}}_{\Pi}, & \overleftrightarrow{\mathcal{F}}_{\Pi}(X) & \triangleq & E \cup X \circledcirc X & (37.4.c)\\
 & = & \mathsf{lfp}^{\subseteq}_{E}\, \widehat{\mathcal{F}}_{\Pi}, & \widehat{\mathcal{F}}_{\Pi}(X) & \triangleq & X \cup X \circledcirc X & (37.4.d)
\end{array}
\qquad \square
$$

Proof of Theorem 37.4 We observe that $\bigcup_{i\in\Delta}(X_i \circledcirc E) = \bigcup_{i\in\Delta}\{\pi x y \mid \pi x \in X_i \wedge \langle x,\ y\rangle \in E\} = \{\pi x y \mid \pi x \in \bigcup_{i\in\Delta} X_i \wedge \langle x,\ y\rangle \in E\} = (\bigcup_{i\in\Delta} X_i) \circledcirc E$ so that the transformer $\overrightarrow{\mathcal{F}}_{\Pi}$ preserves non-empty joins so is upper continuous. Same for $\overleftarrow{\mathcal{F}}_{\Pi}$.

Let $\langle X_i,\ i \in \mathbb{N}\rangle$ be a $\subseteq$-increasing chain of elements of $\wp(V^{>1})$. $\circledcirc$ is componentwise increasing so $\bigcup_{i\in\mathbb{N}}(X_i \circledcirc X_i) \subseteq (\bigcup_{i\in\mathbb{N}} X_i \circledcirc \bigcup_{i\in\mathbb{N}} X_i)$. Conversely if $\pi \in (\bigcup_{i\in\mathbb{N}} X_i \circledcirc \bigcup_{i\in\mathbb{N}} X_i)$ then $\pi = \pi_i x \pi_j$ where $\pi_i x \in X_i$ and $x\pi_j \in X_j$. Assume $i \leqslant j$. Because $X_i \subseteq X_j$, $\pi_i x \in X_j$ so $\pi = \pi_i x \pi_j \in X_j \circledcirc X_j \subseteq \bigcup_{k\in\mathbb{N}} X_k \circledcirc X_k$ proving that $\bigcup_{i\in\mathbb{N}}(X_i \circledcirc X_i) \supseteq (\bigcup_{i\in\mathbb{N}} X_i \circledcirc \bigcup_{i\in\mathbb{N}} X_i)$. We conclude, by antisymmetry, that $\overleftrightarrow{\mathcal{F}}_{\Pi}$ and $\widehat{\mathcal{F}}_{\Pi}$ are upper continuous.

It follows, by Kleene-Scott's fixpoint Theorem 13.23, that the least fixpoints do exist.

We consider case (37.4.c). By upper continuity, we can apply Theorem 16.20. Let us calculate the iterates $\langle \overleftrightarrow{\mathcal{F}}_{\Pi}^{\,k},\ k \in \mathbb{N}\rangle$ of the fixpoint of transformer $\overleftrightarrow{\mathcal{F}}_{\Pi}$ from $\varnothing$.

$\overleftrightarrow{\mathcal{F}}_{\Pi}^{\,0} = \varnothing$, by def. of the iterates from $\varnothing$.

$\overleftrightarrow{\mathcal{F}}_{\Pi}^{\,1} = \overleftrightarrow{\mathcal{F}}_{\Pi}(\overleftrightarrow{\mathcal{F}}_{\Pi}^{\,0}) = E = \Pi^2(G)$ contains the paths of length 1 which are made of a single arc. If the graph has no paths longer than mere arcs, all paths are covered after 1 iteration.

Assume, by recurrence hypothesis on k, that $\overleftrightarrow{\mathcal{F}}_{\Pi}^{\,k} = \bigcup_{n=2}^{2^{k-1}} \Pi^n(G)$ contains exactly all paths of G of length less than or equal to 2^{k-1}. We have

$$
\begin{array}{rll}
\overleftrightarrow{\mathcal{F}}_{\Pi}^{\,k+1} & \triangleq \overleftrightarrow{\mathcal{F}}_{\Pi}(\overleftrightarrow{\mathcal{F}}_{\Pi}^{\,k}) & \text{(def. iterates)}\\
 & = E \cup \overleftrightarrow{\mathcal{F}}_{\Pi}^{\,k} \circledcirc \overleftrightarrow{\mathcal{F}}_{\Pi}^{\,k} & \text{(def. } \overleftrightarrow{\mathcal{F}}_{\Pi}\text{)}\\
 & = E \cup \{x_1 \dots x_n y_2 \dots y_m \mid x_1 \dots x_n \in \overleftrightarrow{\mathcal{F}}_{\Pi}^{\,k} \wedge x_n y_2 \dots y_m \in \overleftrightarrow{\mathcal{F}}_{\Pi}^{\,k}\} & \text{(def. } \circledcirc\text{)}\\
 & = E \cup \{x_1 \dots x_n y_2 \dots y_m \mid x_1 \dots x_n \in \displaystyle\bigcup_{n=2}^{2^{k-1}} \Pi^n(G) \wedge x_n y_2 \dots y_m \in \bigcup_{n=2}^{2^{k-1}} \Pi^n(G)\} & \\
 & & \text{(ind. hyp.)}\\
 & = E \cup \displaystyle\bigcup_{n=3}^{2^k} \Pi^n(G) &
\end{array}
$$

(($\subseteq$) the concatenation of two path of length at least 1 and at most 2^{k-1} is at least of length 2 and at most of length $2 \times 2^{k-1} = 2^k$.
($\supseteq$) Conversely, any path of length at most 2^k has either length 1 in E or can be decomposed into two paths $\pi = x_1 \dots x_n$ and $\pi' = x_n y_2 \dots y_m$ of length at most 2^{k-1}. By induction hypothesis, $\pi, \pi' \in \bigcup_{n=2}^{2^{k-1}} \Pi^n(G)$)

By recurrence on k, for all $k \in \mathbb{N}^+$, $\overleftrightarrow{\mathscr{F}}_{\Pi}{}^{k} = \bigcup_{n=2}^{2^{k-1}} \Pi^n(G)$ contains exactly all paths from x to y of length less than or equal to 2^{k-1}.

Finally, we must prove that the limit $\mathsf{lfp}^{\subseteq}\, \overleftrightarrow{\mathscr{F}}_{\Pi} = \bigcup_{k\in\mathbb{N}} \overleftrightarrow{\mathscr{F}}_{\Pi}{}^{k}$ is $\Pi(G)$ that is contains exactly all paths of G.

Any path in $\Pi(G)$ has a length $n > 0$ such that $n \leqslant 2^{n-1}$ so belongs to $\overleftrightarrow{\mathscr{F}}_{\Pi}{}^{n}(\varnothing)$ hence to the limit, proving $\Pi(G) \subseteq \mathsf{lfp}^{\subseteq}\, \overleftrightarrow{\mathscr{F}}_{\Pi}$.

Conversely any path in $\mathsf{lfp}^{\subseteq}\, \overleftrightarrow{\mathscr{F}}_{\Pi} = \bigcup_{k\in\mathbb{N}} \overleftrightarrow{\mathscr{F}}_{\Pi}{}^{k}$ belongs to some iterate $\overleftrightarrow{\mathscr{F}}_{\Pi}{}^{k}$ which contains exactly all paths of length less than or equal to 2^k so belongs to $\Pi^{2^k}(G)$ hence to $\Pi(G)$, proving $\mathsf{lfp}^{\subseteq}\, \overleftrightarrow{\mathscr{F}}_{\Pi} \subseteq \Pi(G)$. By antisymmetry $\Pi(G) = \mathsf{lfp}^{\subseteq}\, \overleftrightarrow{\mathscr{F}}_{\Pi}$.

The equivalent form $\widehat{\mathscr{F}}_{\Pi}$ follows from Exercises 13.7 and 13.8. The proofs for (37.4.a,b) are similar.

$\square$

Exercise 37.5. Show that in cases (37.4.a) and (37.4.b), the k-iterate is $\bigcup_{n=2}^{k} \Pi^n(G)$. $\square$

37.4 Abstraction of the paths of a graph

A *path problem* in a graph $G = \langle V,\ E\rangle$ consists in specifying/computing an abstraction $\alpha(\Pi(G))$ of its paths $\Pi(G)$ defined by a Galois connection

$$\langle \wp(V^{>1}),\ \subseteq,\ \cup\rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle A,\ \sqsubseteq,\ \sqcup\rangle.$$

A path problem can be solved by a fixpoint definition/computation.

Theorem 37.6 (Fixpoint characterization of a path problem) Let $G = \langle V,\ E\rangle$ be a graph with paths $\Pi(G)$ and $\langle \wp(V^{>1}),\ \subseteq,\ \cup\rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle A,\ \sqsubseteq,\ \sqcup\rangle$.

$$\begin{aligned}
\alpha(\Pi(G)) &= \mathsf{lfp}^{\sqsubseteq}\, \overrightarrow{\mathscr{F}}^{\sharp}_{\Pi}, & \overrightarrow{\mathscr{F}}^{\sharp}_{\Pi}(X) &\triangleq \alpha(E) \sqcup X \mathbin{\overline{\circledcirc}} \alpha(E) && (37.6.a)\\
&= \mathsf{lfp}^{\sqsubseteq}\, \overleftarrow{\mathscr{F}}^{\sharp}_{\Pi}, & \overleftarrow{\mathscr{F}}^{\sharp}_{\Pi}(X) &\triangleq \alpha(E) \sqcup \alpha(E) \mathbin{\overline{\circledcirc}} X && (37.6.b)\\
&= \mathsf{lfp}^{\sqsubseteq}\, \overleftrightarrow{\mathscr{F}}^{\sharp}_{\Pi}, & \overleftrightarrow{\mathscr{F}}^{\sharp}_{\Pi}(X) &\triangleq \alpha(E) \sqcup X \mathbin{\overline{\circledcirc}} X && (37.6.c)\\
&= \mathsf{lfp}^{\sqsubseteq}_{\alpha(E)}\, \widehat{\mathscr{F}}^{\sharp}_{\Pi}, & \widehat{\mathscr{F}}^{\sharp}_{\Pi}(X) &\triangleq X \sqcup X \mathbin{\overline{\circledcirc}} X && (37.6.d)
\end{aligned}$$

where $\alpha(X \circledcirc Y) = \alpha(X) \mathbin{\overline{\circledcirc}} \alpha(Y)$. $\square$

Proof of Theorem 37.6 All cases are similar. Let us check the commutation for (37.6.c).

$$\begin{aligned}
&\alpha(\overleftrightarrow{\mathscr{F}}_{\Pi}(X))\\
=\ &\alpha(E \cup X \circledcirc X) && \text{(def. (37.4.c) of } \overleftrightarrow{\mathscr{F}}_{\Pi}(X))\\
=\ &\alpha(E) \sqcup \alpha(X \circledcirc X)
\end{aligned}$$

$$
\begin{aligned}
& \qquad \wr\text{by Lemma 11.33, the abstraction of Galois connections preserves existing joins}\wr \\
= {} & \alpha(E) \sqcup \alpha(X) \,\overline{\circledcirc}\, \alpha(X) && \wr\text{by hyp.}\wr \\
= {} & \overleftrightarrow{\mathscr{F}}^{\sharp}_{\Pi}(\alpha(X)) && \wr\text{def. (37.6.c) of } \overleftrightarrow{\mathscr{F}}^{\sharp}_{\Pi}\wr
\end{aligned}
$$

We conclude by Theorem 37.4 and exact least fixpoint abstraction Theorem 16.17. The equivalent form $\widehat{\mathscr{F}}^{\sharp}_{\Pi}$ follows from Exercises 13.7 and 13.8. □

37.5 Calculational design of the paths between any two vertices

As a direct application of Theorem 37.6, let us consider the abstraction of all paths $\Pi(G)$ into the paths between any two vertices. This is $\mathsf{p} \triangleq \alpha^{\circ\!\!-\!\circ}(\Pi(G))$ with the projection abstraction

$$\alpha^{\circ\!\!-\!\circ}(X) \quad \triangleq \quad (y, z) \mapsto \{x_1 \dots x_n \in X \mid y = x_1 \wedge x_n = z\}$$

such that

$$\langle \wp(V^{>1}),\ \subseteq,\ \cup\rangle \xrightleftharpoons[\alpha^{\circ\!\!-\!\circ}]{\gamma^{\circ\!\!-\!\circ}} \langle V \times V \to \wp(V^{>1}),\ \dot{\subseteq},\ \dot{\cup}\rangle \tag{37.7}$$

By (37.2) and Lemma 11.33 on existing join preservation, we have

$$
\begin{aligned}
\mathsf{p}(y, z) \quad & \triangleq \quad \bigcup_{n \in \mathbb{N}^+} \mathsf{p}^n(y, z) && (37.8)\\
\mathsf{p}^n(y, z) \quad & \triangleq \quad \{x_1 \dots x_n \in \Pi^n(G) \mid y = x_1 \wedge x_n = z\} \\
& = \quad \{x_1 \dots x_n \in V^n \mid y = x_1 \wedge x_n = z \wedge \forall i \in [1, n[\ .\ \langle x_i,\ x_{i+1}\rangle \in E\}.
\end{aligned}
$$

$\mathsf{p}(x, x)$ is empty if and only if there is no cycle from x to x (which requires, in particular, that the graph has no self-loops *i.e.* $\forall x \in V\ .\ \langle x,\ x\rangle \notin E$). We define the concatenation of finite paths

$$
\begin{aligned}
x_1 \dots x_n \odot y_1 y_2 \dots y_m \quad & \triangleq \quad x_1 \dots x_n y_2 \dots y_m && \text{if } x_n = y_1 \qquad (37.9)\\
& \triangleq \quad \text{undefined} && \text{otherwise}
\end{aligned}
$$

As a direct application of the path problem Theorem 37.6, we have the following fixpoint characterization of the paths of a graph between any two vertices [4, Sect. 5], which, by Tarski iterative fixpoint Theorem 13.18 and its variants yields an iterative algorithm (converging in finitely many iterations for graphs without infinite paths).

Theorem 37.10 (Fixpoint characterization of the paths of a graph between any two vertices) Let $G = \langle V,\ E\rangle$ be a graph. The paths between any two vertices of G are $\mathsf{p} = \Pi(G)$ such that

$$\begin{array}{rcllcll}
\mathsf{p} & = & \mathsf{lfp}^{\subseteq}\, \overrightarrow{\mathscr{F}}_{\Pi}^{\circ\!-\!\circ}, & \overrightarrow{\mathscr{F}}_{\Pi}^{\circ\!-\!\circ}(\mathsf{p}) & \triangleq & \dot{E} \mathbin{\dot\cup} \mathsf{p} \mathbin{\ddot{\circledcirc}} \dot{E} & (37.10.a)\\
 & = & \mathsf{lfp}^{\subseteq}\, \overleftarrow{\mathscr{F}}_{\Pi}^{\circ\!-\!\circ}, & \overleftarrow{\mathscr{F}}_{\Pi}^{\circ\!-\!\circ}(\mathsf{p}) & \triangleq & \dot{E} \mathbin{\dot\cup} \dot{E} \mathbin{\ddot{\circledcirc}} \mathsf{p} & (37.10.b)\\
 & = & \mathsf{lfp}^{\subseteq}\, \overleftrightarrow{\mathscr{F}}_{\Pi}^{\circ\!-\!\circ}, & \overleftrightarrow{\mathscr{F}}_{\Pi}^{\circ\!-\!\circ}(\mathsf{p}) & \triangleq & \dot{E} \mathbin{\dot\cup} \mathsf{p} \mathbin{\ddot{\circledcirc}} \mathsf{p} & (37.10.c)\\
 & = & \mathsf{lfp}^{\subseteq}_{\dot{E}}\, \widehat{\mathscr{F}}_{\Pi}^{\circ\!-\!\circ}, & \widehat{\mathscr{F}}_{\Pi}^{\circ\!-\!\circ}(\mathsf{p}) & \triangleq & \mathsf{p} \mathbin{\dot\cup} \mathsf{p} \mathbin{\ddot{\circledcirc}} \mathsf{p} & (37.10.d)
\end{array}$$

where $\dot{E} \triangleq x, y \mapsto (E \cap \{\langle x,\ y\rangle\})$ and $\mathsf{p}_1 \mathbin{\ddot{\circledcirc}} \mathsf{p}_2 \triangleq x, y \mapsto \bigcup_{z\in V} \mathsf{p}_1(x,z) \circledcirc \mathsf{p}_2(z,y)$. □

Proof of Theorem 37.10 We apply Theorem 37.6 with $\alpha^{\circ\!-\!\circ}(E) = x, y \mapsto (E \cap \{\langle x,\ y\rangle\}) = \dot{E}$ and

$$\begin{aligned}
&\alpha^{\circ\!-\!\circ}(X \circledcirc Y)\\
=\ & (x,y) \mapsto \{z_1 \ldots z_n \in X \circledcirc Y \mid x = z_1 \wedge z_n = y\} && \text{(def. (37.7) of } \alpha^{\circ\!-\!\circ}\text{)}\\
=\ & (x,y) \mapsto \{z_1 \ldots z_n \in \{x_1 \ldots x_k y_2 \ldots y_m \mid x_1 \ldots x_k \in X \wedge x_k y_2 \ldots y_m \in Y\} \mid x = z_1 \wedge z_n = y\}\\
& && \text{(def. (37.3) of } \circledcirc\text{)}\\
=\ & (x,y) \mapsto \bigcup_{z\in V}\{x x_2 \ldots x_{k-1} z y_2 \ldots, y_{m-1} y \mid x x_2 \ldots x_{k-1} z \in X \wedge z y_2 \ldots y_{m-1} y \in Y\}\\
& && \text{(def. } \in \text{ and } \cup \text{ with } x = x_1,\ y_m = y, \text{ and } z = x_k\text{)}\\
=\ & (x,y) \mapsto \bigcup_{z\in V}\{x x_2 \ldots x_{k-1} z \odot z y_2 \ldots y_{m-1} y \mid x x_2 \ldots x_{k-1} z \in X \wedge z y_2 \ldots y_{m-1} y \in Y\}\\
& && \text{(def. (37.9) of } \odot\text{)}\\
=\ & (x,y) \mapsto \bigcup_{z\in V}\{p \odot p' \mid p \in \alpha^{\circ\!-\!\circ}(X)(y,z) \wedge p' \in \alpha^{\circ\!-\!\circ}(Y)(z,y)\}\\
& && \text{(def. } \alpha^{\circ\!-\!\circ}(X) \text{ with } p = x x_2 \ldots x_{k-1} z \text{ and } p' = z y_2 \ldots y_{m-1} y\text{)}\\
=\ & \alpha^{\circ\!-\!\circ}(X) \mathbin{\ddot{\circledcirc}} \alpha^{\circ\!-\!\circ}(Y)
\end{aligned}$$

by defining $X \mathbin{\ddot{\circledcirc}} Y \triangleq (x,y) \mapsto \bigcup_{z\in V}\{p \odot p' \mid p \in X(y,z) \wedge p' \in Y(z,y)\} = (x,y) \mapsto \bigcup_{z\in V} X(y,z) \circledcirc Y(z,y)$ by (37.3) and (37.9). □

We now equip graphs with weights *e.g.* to measure the distance between vertices.

37.6 Groups

A *group* $\langle \mathbb{G},\ 0,\ +\rangle$[45] is a set $\mathbb{G}$ with $0 \in \mathbb{G}$ and $+ \in \mathbb{G} \times \mathbb{G} \to \mathbb{G}$ such that

[45] We call group both the algebraic structure $\langle \mathbb{G},\ 0,\ +\rangle$ and its support $\mathbb{G}$.

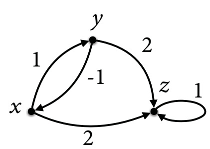
- $\forall a, b, c \in \mathbb{G} \,.\, (a + b) + c = a + (b + c)$ (associativity)
- $\forall a \in \mathbb{G} \,.\, 0 + a = a + 0 = a$ (identity denoted 0)
- $\forall a \in \mathbb{G} \,.\, \exists b \in \mathbb{G} \,.\, a + b = 0$ (inverse, b is denoted $-a$ or a^{-1}.)

For example the scalars $\langle \mathbb{F},\ 0,\ +\rangle$ are a group where 0 is the null scalar and $+$ is scalar addition. Integers modulo n are a group under addition.

37.7 Weighted graphs

Let $\langle \mathbb{G},\ 0,\ +\rangle$ be a group. A (directed) graph $G = \langle V,\ E,\ \boldsymbol{\omega}\rangle$ weighted on the group $\mathbb{G}$ is a finite graph $\langle V,\ E\rangle$ equipped with a weight $\boldsymbol{\omega} \in E \to \mathbb{G}$ mapping arcs to their weight.

Example 37.11 Continuing Example 37.1,



$$\left[\begin{array}{ll} \boldsymbol{\omega}(\langle x,\ y\rangle) = 1 & \boldsymbol{\omega}(\langle x,\ z\rangle) = 2 \\ \boldsymbol{\omega}(\langle x,\ z\rangle) = 2 & \boldsymbol{\omega}(\langle y,\ x\rangle) = -1 \\ \boldsymbol{\omega}(\langle y,\ z\rangle) = 2 & \boldsymbol{\omega}(\langle z,\ z\rangle) = 1 \end{array}\right] \qquad \mathbf{G} = \left[\begin{array}{c|ccc} & x & y & z \\ \hline x & \infty & 1 & 2 \\ y & -1 & \infty & 2 \\ z & \infty & \infty & 1 \end{array}\right]$$

□

37.8 Totally ordered groups

A *totally (or linearly) ordered group* $\langle \mathbb{G},\ \leqslant,\ 0,\ +\rangle$ is a group $\langle \mathbb{G},\ 0,\ +\rangle$ with a total order $\leqslant$ on $\mathbb{G}$ such that

- $\forall a, b, c \in \mathbb{G} \,.\, (a \leqslant b) \Rightarrow (a + c \leqslant b + c)$ (translation-invariance.)

An element $x \in \mathbb{G}$ of a totally ordered group $\langle \mathbb{G},\ \leqslant,\ 0,\ +\rangle$ is said to be strictly negative if and only if $x \leqslant 0 \wedge x \neq 0$.

Following Section 10.3, if $S \subseteq \mathbb{G}$ then we define $\min S$ to be the greatest lower bound of S in $\mathbb{G}$ or $-\infty$:

$$\begin{array}{rcll} \min S &=& m & \Leftrightarrow\quad m \in \mathbb{G} \wedge (\forall x \in S \,.\, m \leqslant x) \wedge (\forall y \in S \,.\, y \leqslant x \Rightarrow y \leqslant m) \\ &=& -\infty & \Leftrightarrow\quad \forall x \in S \,.\, \exists y \in S \,.\, y < x \qquad \text{(where } -\infty \notin \mathbb{G}\text{)} \\ &=& \infty & \Leftrightarrow\quad S = \varnothing \qquad \text{(where } \infty \notin \mathbb{G}\text{)} \end{array}$$

So if $\mathbb{G}$ has no infimum $\min \mathbb{G} = \max \varnothing = -\infty \notin \mathbb{G}$. Similarly, $\max S$ is the least upper bound of S in $\mathbb{G}$. if any, $-\infty$ otherwise, with $\max \mathbb{G} = \min \varnothing = \infty \notin \mathbb{G}$ when $\mathbb{G}$ has no supremum.

Exercise 37.12 (min of a sum). Let $\langle \mathbb{G},\ \leqslant,\ 0,\ +\rangle$ be a totally ordered group, $S_1, S_2 \in \wp(\mathbb{G})$. Extend $+$ with $x + \infty = \infty + x = \infty + \infty = \infty$. Prove that $\min\{x + y \mid x \in S_1 \wedge y \in S_2\} = \min S_1 + \min S_2$. □

37.9 Minimal weight of a set of paths

The weight of a path is

$$\boldsymbol{\omega}(x_1 \dots x_n) \quad \triangleq \quad \sum_{i=1}^{n-1} \boldsymbol{\omega}(\langle x_i,\ x_{i+1}\rangle) \tag{37.13}$$

which is 0 when $n \leqslant 1$ and $\boldsymbol{\omega}(\langle x_1,\ x_2\rangle) + \sum_{i=2}^{n-1} \boldsymbol{\omega}(\langle x_i,\ x_{i+1}\rangle)$ when $n > 1$. The (minimal) weight of a set of paths is

$$\boldsymbol{\omega}(P) \quad \triangleq \quad \min\{\boldsymbol{\omega}(\pi) \mid \pi \in P\}. \tag{37.14}$$

We have $\boldsymbol{\omega}(\bigcup_{i\in\Delta} P_i) = \min_{i\in\Delta} \boldsymbol{\omega}(P_i)$ so a Galois connection

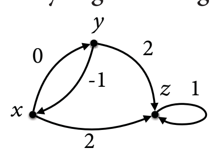
$$\langle \wp(\bigcup_{n\in\mathbb{N}^+} V^{\,n}),\ \subseteq\rangle \xleftrightarrows[\boldsymbol{\omega}]{} \langle \mathbb{G} \cup \{-\infty, \infty\},\ \geqslant\rangle$$

and the complete lattice $\langle \mathbb{G} \cup \{-\infty, \infty\},\ \geqslant,\ \infty,\ -\infty,\ \min,\ \max\rangle$.

Extending pointwise to $V \times V \to \wp(\bigcup_{n\in\mathbb{N}^+} V^{\,n})$ with $\dot{\boldsymbol{\omega}}(\mathsf{p})\langle x,\ y\rangle \triangleq \boldsymbol{\omega}(\mathsf{p}(x, y))$, $\mathsf{d} \mathrel{\dot{\leqslant}} \mathsf{d}' \triangleq \forall x, y\ .\ \mathsf{d}\langle x,\ y\rangle \leqslant \mathsf{d}'\langle x,\ y\rangle$, and $\dot{\geqslant}$ is the inverse of $\dot{\leqslant}$, we have

$$\langle V \times V \to \wp(\bigcup_{n\in\mathbb{N}^+} V^{\,n}),\ \dot{\subseteq}\rangle \xleftrightarrows[\dot{\boldsymbol{\omega}}]{} \langle V \times V \to \mathbb{G} \cup \{-\infty, \infty\},\ \dot{\geqslant}\rangle. \tag{37.15}$$

Example 37.16 (Graph with cycle of strictly negative weight) The following graph



has a cycle x, y, x of weight -1, x, y, x, y, x of weight -2, x, y, x, y, x, y, x of weight -3, *etc.* so that the minimum distance between x and y is $-\infty$. □

37.10 Shortest distance along the graph paths

The distance $\mathsf{d}(x, y)$ between an origin $x \in V$ and an extremity $y \in V$ of a weighted finite graph $G = \langle V,\ E,\ \boldsymbol{\omega}\rangle$ on a totally ordered group $\langle \mathbb{G},\ \leqslant,\ 0,\ +\rangle$ is the length $\boldsymbol{\omega}(\mathsf{p}(x, y))$ of the shortest path between these vertices

$$\mathsf{d} \quad \triangleq \quad \dot{\boldsymbol{\omega}}(\mathsf{p})$$

where p has a fixpoint characterization given by Theorem 37.10.

37.11 Calculational design of the shortest distances between any two vertices

The shortest distance between vertices of a weighted graph is a path problem solved by Theorem 37.6, the composition of the abstractions and (37.15) and (37.7), and the path abstraction Theorem 37.6. Theorem 37.17 is based on (37.6.d), (37.6.a—c) provide three other solutions.

Theorem 37.17 (Fixpoint characterization of the shortest distances of a graph) Let $G = \langle V, E, \boldsymbol{\omega}\rangle$ be a graph weighted on the totally ordered group $\langle \mathbb{G}, \leqslant, 0, +\rangle$. Then the distances between any two vertices are

$$\mathsf{d} \;=\; \dot{\boldsymbol{\omega}}(\mathsf{p}) \;=\; \mathsf{gfp}^{\dot{\leqslant}}_{E^{\boldsymbol{\omega}}}\, \widehat{\mathscr{F}}^{\delta}_{G} \qquad \text{where} \tag{37.17}$$
$$E^{\boldsymbol{\omega}} \;\triangleq\; (x, y) \mapsto [\![\, \langle x,\ y\rangle \in E \mathrel{?} \boldsymbol{\omega}(x, y) \mathrel{:} \infty \,]\!]$$
$$\widehat{\mathscr{F}}^{\delta}_{G}(X) \;\triangleq\; (x, y) \mapsto\ \min\{X(x, y), \min_{z\in V}\{X(x, z) + X(z, y)\}\} \qquad \square$$

Proof of Theorem 37.17 We apply Theorem 37.6 with abstraction $\dot{\boldsymbol{\omega}} \circ \alpha^{\circ\!\!-\!\!\circ}$ so that we have to abstract the transformers in Theorem 37.10 using an exact fixpoint abstraction of Theorem 16.17. The initialization and commutation condition yield the transformers by calculational design.

– $\dot{\boldsymbol{\omega}} \circ \alpha^{\circ\!\!-\!\!\circ}(E)(x, y)$

$= \boldsymbol{\omega}(E \cap \{\langle x,\ y\rangle\})$ (as proved for Theorem 37.10 and def. $\dot{\boldsymbol{\omega}}$)

$= [\![\, \langle x,\ y\rangle \in E \mathrel{?} \boldsymbol{\omega}(x, y) \mathrel{:} \min \varnothing \,]\!]$ (def. $\cap$, conditional, and $\boldsymbol{\omega}$)

$= [\![\, \langle x,\ y\rangle \in E \mathrel{?} \boldsymbol{\omega}(x, y) \mathrel{:} \infty \,]\!]$ (def. min)

– $\dot{\boldsymbol{\omega}} \circ \alpha^{\circ\!\!-\!\!\circ}(X \circledcirc Y)(x, y)$

$= \dot{\boldsymbol{\omega}}(\alpha^{\circ\!\!-\!\!\circ}(X) \mathbin{\ddot{\circledcirc}} \alpha^{\circ\!\!-\!\!\circ}(Y))(x, y)$ (as proved for Theorem 37.10)

$= \boldsymbol{\omega}(\alpha^{\circ\!\!-\!\!\circ}(X) \mathbin{\ddot{\circledcirc}} \alpha^{\circ\!\!-\!\!\circ}(Y))(x, y))$ (pointwise def. (37.15) of $\dot{\boldsymbol{\omega}}$)

$= \boldsymbol{\omega}(\bigcup_{z\in V} \alpha^{\circ\!\!-\!\!\circ}(X)(x, z) \circledcirc \alpha^{\circ\!\!-\!\!\circ}(Y)(z, y)))$ (def. $\ddot{\circledcirc}$ in Theorem 37.10)

$= \min_{z\in V} \boldsymbol{\omega}(\alpha^{\circ\!\!-\!\!\circ}(X)(x, z) \circledcirc \alpha^{\circ\!\!-\!\!\circ}(Y)(z, y)))$ (Galois connection (37.14))

$= \min_{z\in V} \boldsymbol{\omega}(\{x_1 \dots x_n y_2 \dots y_m \mid x_1 \dots x_n \in \alpha^{\circ\!\!-\!\!\circ}(X)(x, z) \wedge x_n y_2 \dots y_m \in \alpha^{\circ\!\!-\!\!\circ}(Y)(z, y)\})$
(def. (37.3) of $\circledcirc$)

$= \min_{z\in V}\{\boldsymbol{\omega}(x_1 \dots x_n y_2 \dots y_m) \mid x_1 \dots x_n \in \alpha^{\circ\!\!-\!\!\circ}(X)(x, z) \wedge x_n y_2 \dots y_m \in \alpha^{\circ\!\!-\!\!\circ}(Y)(z, y)\})$
(def. (37.14) of $\boldsymbol{\omega}$)

$= \min_{z\in V}\{\boldsymbol{\omega}(x_1 \dots x_n) + \boldsymbol{\omega}(x_n y_2 \dots y_m) \mid x_1 \dots x_n \in \alpha^{\circ\!\!-\!\!\circ}(X)(x, z) \wedge x_n y_2 \dots y_m \in \alpha^{\circ\!\!-\!\!\circ}(Y)(z, y)\})$
(def. (37.13) of $\boldsymbol{\omega}$)

$= \min_{z\in V}\{\boldsymbol{\omega}(x_1 \dots x_n) + \boldsymbol{\omega}(y_1 y_2 \dots y_m) \mid x_1 \dots x_n \in \alpha^{\circ\!\!-\!\!\circ}(X)(x, z) \wedge y_1 y_2 \dots y_m \in \alpha^{\circ\!\!-\!\!\circ}(Y)(z, y)\})$
(def. $\alpha^{\circ\!\!-\!\!\circ}$ so that $x_1 = x$, $x_n = y_1 = z$, and $y_m = y$)

$= \min_{z\in V} \min\{\boldsymbol{\omega}(x_1 \dots x_n) \mid x_1 \dots x_n \in \alpha^{\circ\!\!-\!\!\circ}(X)(x, z)\} + \min\{\boldsymbol{\omega}(y_1 y_2 \dots y_m) \mid y_1 y_2 \dots y_m \in \alpha^{\circ\!\!-\!\!\circ}(Y)(z, y)\}$ (min of a sum in Exercise 37.12)

$= \min_{z\in V} \min\{\boldsymbol{\omega}(\pi_1) \mid \pi_1 \in \alpha^{\circ\!\!-\!\!\circ}(X)(x, z)\} + \min\{\boldsymbol{\omega}(\pi_2) \mid \pi_2 \in \alpha^{\circ\!\!-\!\!\circ}(Y)(z, y)\}$
(letting $\pi_1 = x_1 \dots x_n$ and $\pi_2 = y_1 y_2 \dots y_m$)

$$\begin{aligned}
&= \min_{z\in V} \boldsymbol{\omega}(\alpha^{\circ\!\!-\!\!\circ}(X)(x,z)) + \boldsymbol{\omega}(\alpha^{\circ\!\!-\!\!\circ}(Y)(z,y)) && \text{\{def. (37.14) of } \boldsymbol{\omega}\text{\}}\\
&= \min_{z\in V} \dot{\boldsymbol{\omega}} \circ \alpha^{\circ\!\!-\!\!\circ}(X)(x,z) + \dot{\boldsymbol{\omega}} \circ \alpha^{\circ\!\!-\!\!\circ}(Y)(z,y) && \text{\{pointwise def. (37.15) of } \dot{\boldsymbol{\omega}}\text{\}}
\end{aligned}$$

By Theorem 37.6 and (37.10.d), we get the transformer $\widehat{\mathscr{F}}_G^{\delta}$. □

Of course the greatest fixpoint in Theorem 37.17 is not computable for infinite graphs.

Exercise 37.18. Calculate shortest distances between vertices of the infinite weighted graph $\langle \mathbb{N}, \{\langle n,\ n+1\rangle \mid n \in \mathbb{N}\},\ \langle n,\ m\rangle \mapsto 1\rangle$. □

For finite graphs, there is a problem with cycles with strictly negative weights. As shown by Example 37.16, the minimal distance between the extremities x and y of the cycle xyx is $-\infty$. It is obtained as the limit of an infinite iteration for the greatest fixpoint in Theorem 37.17. Following Roy-Floyd-Warshall, we will assume that the graph has no cycle with negative weight in which case the iterative computation of the greatest fixpoint in Theorem 37.17 does converge to the shortest distance between any two vertices.

For a finite graph of n vertices, the computation of $\mathsf{gfp}^{\dot{\leqslant}}_{E^{\boldsymbol{\omega}}}\ \widehat{\mathscr{F}}_G^{\delta}$ in (37.17) has to consider all pairs of vertices in n^2, for each such pair $\langle x,\ y\rangle$ the n vertices $z \in V$, and n iterations may be necessary to converge along a path going through all vertices, so would be in $O(n^4)$.

However, the iteration in Roy-Floyd-Warshall algorithm is much more efficient in $O(n^3)$, since it does not consider all paths in the graph but only simple paths that over-approximate paths with no cycles (called elementary paths). Let us design the Roy-Floyd-Warshall algorithm by calculus.

37.12 Elementary paths and cycles

A cycle is *elementary* if and only if it contains no internal subcycle (*i.e.* subpath which is a cycle). A path is *elementary* if and only if it contains no subpath which is an internal cycle (so an elementary cycle is an elementary path). The only vertices that can have two occurrences in an elementary path are its extremities in which case it is an elementary cycle.

Example 37.19 In Example 37.1, xyz and xz are elementary paths (zz is a path of length 1). The path xyx is an elementary cycle (zz is a cycle of length 1). The paths $xyzz$ and $xyxyxz$ and the cycles $yxyxy$ and zzz are not elementary. We do not consider infinite paths such as $xyxyx\ldots$. □

Lemma 37.20 (elementary path) A path $x_1 \ldots x_n \in \Pi^n(G)$ is elementary if and only if

$$\begin{aligned}
\mathsf{elem?}(x_1 \ldots x_n) \quad &\triangleq \quad (\forall i, j \in [1, n] \,.\, (i \neq j) \Rightarrow (x_i \neq x_j)) \vee && (37.20)\\
&\qquad (x_1 = x_n \wedge \mathsf{elem?}(x_1 \ldots x_{n-1})) && \text{(case of a cycle)}
\end{aligned}$$

is true. □

Proof of Lemma 37.20 — The necessary condition ($x_1 \ldots x_n \in \Pi^n(G)$ is elementary implies that $\mathsf{elem?}(x_1 \ldots x_n)$) is proved contrapositively.

$$\begin{aligned}
&\neg(\mathsf{elem?}(x_1 \ldots x_n)) \\
=\ &\neg((\forall i, j \in [1, n] \,.\, (i \neq j) \Rightarrow (x_i \neq x_j)) \vee (x_1 = x_n \wedge \mathsf{elem?}(x_1 \ldots x_n))) && \text{⟅def. elem?⟆} \\
=\ &(\exists i, j \in [1, n] \,.\, i \neq j \wedge x_i = x_j) \wedge ((x_1 = x_n) \Rightarrow (\exists i, j \in [1, n[\,.\, i \neq j \wedge x_i \neq x_j)) \\
& && \text{⟅De Morgan laws⟆}
\end{aligned}$$

By $\exists i, j \in [1, n] \,.\, i \neq j \wedge x_i = x_j$ the path $x_1 \ldots x_n$ must have a cycle, but this is not forbidden if $x_1 = x_n$. In that case, the second condition $(x_1 = x_n) \Rightarrow (\exists i, j \in [1, n[\,.\, i \neq j \wedge x_i \neq x_j)$ implies that there is a subcycle within $x_1 \ldots x_{n-1}$, so the cycle $x_1 \ldots x_{n-1}x_1$ is not elementary.

— Conversely, the sufficient condition ($\mathsf{elem?}(x_1 \ldots x_n) \Rightarrow x_1 \ldots x_n \in \Pi^n(G)$ is elementary) is proved by *reductio ad absurdum*. Assume $\mathsf{elem?}(x_1 \ldots x_n)$ and $x_1 \ldots x_n \in \Pi^n(G)$ is not elementary so has an internal subcycle.

- If $x_1 = x_n$, the internal subcycle is $x_1 \ldots x_{n-1} = \pi_1 a \pi_2 a \pi_3$ so $\exists i, j \in [1, n[\,.\, i \neq j \wedge x_i \neq x_j$ in contradiction with $\mathsf{elem?}(x_1 \ldots x_n)$.
- Otherwise $x_1 \neq x_n$ and the internal subcycle has the form $x_1 \ldots x_n = \pi_1 a \pi_2 a \pi_3$ where, possibly $\pi_1 a = x_1$ or $a\pi_3 = x_n$, but not both, so $\exists i, j \in [1, n] \,.\, i \neq j \wedge x_i \neq x_j$ in contradiction with $\mathsf{elem?}(x_1 \ldots x_n)$. □

37.13 Calculational design of the elementary paths between any two vertices

Restricting paths to elementary ones is the abstraction

$$\alpha^{\partial}(P) \quad \triangleq \quad \{\pi \in P \mid \mathsf{elem?}(\pi)\}.$$

Notice that, by (37.20), cycles (such as x, x for a self-loop $\langle x,\ x\rangle \in E$) are not excluded, provided it is through the path extremities. By Exercise 11.2, this exclusion abstraction is a Galois connection.

$$\langle \wp(V^{>1}),\ \subseteq\rangle \underset{\alpha^{\partial}}{\overset{\gamma^{\partial}}{\leftrightarrows}} \langle \wp(V^{>1}),\ \subseteq\rangle$$

which extends pointwise to

$$\langle V \times V \to \wp(V^{>1}),\ \dot{\subseteq}\rangle \underset{\dot{\alpha}^{\partial}}{\overset{\dot{\gamma}^{\partial}}{\leftrightarrows}} \langle V \times V \to \wp(V^{>1}),\ \dot{\subseteq}\rangle$$

The following Lemma 37.21 provides a necessary and sufficient condition for the concatenation of two elementary paths to be elementary.

Lemma 37.21 (concatenation of elementary paths) If $x\pi_1 z$ and $z\pi_2 y$ are elementary paths then their concatenation $\pi_1 \odot \pi_2 = x\pi_1 z\pi_2 y$ is elementary if and only if

$$\begin{aligned}
\mathsf{elem\text{-}conc?}(x\pi_1 z, z\pi_2 y) \quad \triangleq \quad & (\mathsf{V}(x\pi_1 z) \cap \mathsf{V}(\pi_2 y) = \varnothing) \vee \\
& (x = y \neq z \wedge \mathsf{V}(\pi_1 z) \cap \mathsf{V}(\pi_2) = \varnothing)
\end{aligned} \tag{37.21}$$

is true. □

Example 37.22 Assume x, y, z, a, and b are all different vertices. $xaz \odot zbx = xazbx$ of the form $x\pi_1 z \odot z\pi_2 y$ with $x = y \neq z \wedge \mathsf{V}(az) \cap \mathsf{V}(b) = \varnothing$ is elementary. $xyz \odot zay = xyzay$ with $x \neq y \wedge \mathsf{V}(xyz) \cap \mathsf{V}(ay) = \{y\} \neq \varnothing$ is not elementary. $xx \odot xx = xxx$ and $xax \odot xbx = xaxbx$ of the form $x\pi_1 z \odot z\pi_2 y$ with $x = z = y$ are not elementary. $xz \odot zz = xzz$ of the form $x\pi_1 z \odot z\pi_2 y$ with $y = z$ is not. □

Proof of Lemma 37.21 Assuming $x\pi_1 z$ and $z\pi_2 y$ to be elementary, we must prove that elem-conc?(π_1, π_2) is true $\Leftrightarrow \pi_1 \odot \pi_2$ is elementary.

— We prove the necessary condition ($\pi_1 \odot \pi_2$ is elementary $\Rightarrow$ elem-conc?(π_1, π_2)) by contraposition ($\neg$elem-conc?(π_1, π_2) $\Rightarrow \pi_1 \odot \pi_2$ has an internal cycle). We have

$$\neg((\mathsf{V}(x\pi_1 z) \cap \mathsf{V}(\pi_2 y) = \varnothing) \vee (x = y \wedge x \neq z \wedge y \neq z \wedge \mathsf{V}(\pi_1 z) \cap \mathsf{V}(\pi_2) = \varnothing))$$
$$= ((\mathsf{V}(x\pi_1 z) \cap \mathsf{V}(\pi_2 y) \neq \varnothing) \wedge (x \neq y \vee x = z \vee y = z \vee \mathsf{V}(\pi_1 z) \cap \mathsf{V}(\pi_2) \neq \varnothing)) \quad \text{\{de Morgan laws\}}$$

- If $x = y$ then $x \in (\mathsf{V}(x\pi_1 z) \cap \mathsf{V}(\pi_2 y)) \neq \varnothing$ so
  - Either $x = z$ or $y = z$ and $\pi_1 \odot \pi_2 = x\pi_1 z\pi_2 y = x\pi_1 x\pi_2 x$ has two internal cycles $x\pi_1 x$ and $x\pi_2 x$ so, by (37.20), is not elementary;
  - Or $\mathsf{V}(\pi_1 z) \cap \mathsf{V}(\pi_2) \neq \varnothing$ with
    - either $\mathsf{V}(\pi_1) \cap \mathsf{V}(\pi_2) \neq \varnothing$ so $\pi_1 = \pi_1' a \pi_1''$ and $\pi_2 = \pi_2' a \pi_2''$ and therefore $\pi_1 \odot \pi_2 = x\pi_1 z\pi_2 y = x\pi_1' a \pi_1'' z \pi_2' a \pi_2'' x$ has an internal cycle $a\pi_1'' z\pi_2' a$,
    - or $z \in \mathsf{V}(\pi_2)$ so $\pi_2 = \pi_2' z \pi_2''$ and therefore $\pi_1 \odot \pi_2 = x\pi_1 z\pi_2 y = x\pi_1 z\pi_2' z\pi_2'' x$ has an internal cycle $z\pi_2' z$;
- Otherwise $x \neq y$ and we have $\mathsf{V}(x\pi_1 z) \cap \mathsf{V}(\pi_2 y) \neq \varnothing$. By cases.
  - If x appears in $\pi_2 y$ that is in π_2 since $x \neq y$ we have $\pi_2 = \pi_2' x \pi_2''$ and then $\pi_1 \odot \pi_2 = x\pi_1 z\pi_2 y = x\pi_1 z\pi_2' x\pi_2'' y$ has an internal cycle $x\pi_1 z\pi_2' x$;
  - Else, if $\mathsf{V}(\pi_1) \cap \mathsf{V}(\pi_2 y) \neq \varnothing$ then
    - Either $\mathsf{V}(\pi_1) \cap \mathsf{V}(\pi_2) \neq \varnothing$ so $\pi_1 = \pi_1' a \pi_1''$ and $\pi_2 = \pi_2' a \pi_2''$ and therefore $\pi_1 \odot \pi_2 = x\pi_1 z\pi_2 y = x\pi_1' a\pi_1'' z\pi_2' a\pi_2'' x$ has an internal cycle $a\pi_1'' z\pi_2' a$,
    - Or $y \in \mathsf{V}(\pi_1)$ so $\pi_1 = \pi_1' y \pi_1''$ and then $\pi_1 \odot \pi_2 = x\pi_1 z\pi_2 y = x\pi_1' y\pi_1'' z\pi_2 y$ has an internal cycle $y\pi_1'' z\pi_2 y$;
  - Otherwise, $z \in \mathsf{V}(\pi_2 y) \neq \varnothing$ and then
    - Either $z \in \mathsf{V}(\pi_2)$ so $\pi_2 = \pi_2' z \pi_2''$ and $\pi_1 \odot \pi_2 = x\pi_1 z\pi_2 y = x\pi_1 z\pi_2' z\pi_2'' y$ has an internal cycle $z\pi_2' z$,
    - Or $z = y$ and $\pi_1 \odot \pi_2 = x\pi_1 z\pi_2 y = x\pi_1 z\pi_2 z$ has an internal cycle $z\pi_2 z$.

— We prove that the condition is sufficient (elem-conc?(π_1, π_2) $\Rightarrow \pi_1 \odot \pi_2$ is elementary) by *reductio ad absurdum*. Assume $x\pi_1 z$, and $z\pi_2 y$ are elementary, elem-conc?($x\pi_1 z, z\pi_2 y$) holds, but that $x\pi_1 z \odot z\pi_2 y = x\pi_1 z\pi_2 y$ is not elementary. By hypothesis, the internal cycle can neither be in $x\pi_1 z$ nor in $z\pi_2 y$ so $\mathsf{V}(x\pi_1 z) \cap \mathsf{V}(\pi_2 y) \neq \varnothing$. Since elem-conc?($\pi_1, \pi_2$) holds, it follows that

$x = y \neq z \wedge \mathsf{V}(\pi_1 z) \cap \mathsf{V}(\pi_2) = \varnothing$ in contradiction with the existence of an internal cycle $a\pi''\pi_2'a$ requiring $\pi_1 z = \pi' a \pi''$ and $\pi_2 = \pi_2' a \pi_2''$ so $a \in \mathsf{V}(\pi' a \pi'') \cap \mathsf{V}(\pi_2' a \pi_2'') = \mathsf{V}(\pi_1 z) \cap \mathsf{V}(\pi_2) \neq \varnothing$. $\square$

We have the following fixpoint characterization of the elementary paths of a graph (converging in finitely many iterations for graphs without infinite paths).

Theorem 37.23 (Fixpoint characterization of the elementary paths of a graph) Let $G = \langle V, E\rangle$ be a graph. The elementary paths between any two vertices of G are $\mathsf{p}^{\backepsilon} \triangleq \alpha^{\circ\!-\!\circ} \circ \alpha^{\backepsilon}(\Pi(G))$ such that

$$
\begin{array}{rclrcll}
\mathsf{p}^{\backepsilon} & = & \mathsf{lfp}^{\dot{\subseteq}}\, \overrightarrow{\mathscr{F}}^{\backepsilon}_{\Pi}, & \overrightarrow{\mathscr{F}}^{\backepsilon}_{\Pi}(\mathsf{p}) & \triangleq & \dot{E} \,\dot{\cup}\, \mathsf{p} \,\dot{\circledcirc}^{\backepsilon}\, \dot{E} & (37.23.\mathrm{a}) \\
 & = & \mathsf{lfp}^{\dot{\subseteq}}\, \overleftarrow{\mathscr{F}}^{\backepsilon}_{\Pi}, & \overleftarrow{\mathscr{F}}^{\backepsilon}_{\Pi}(\mathsf{p}) & \triangleq & \dot{E} \,\dot{\cup}\, \dot{E} \,\dot{\circledcirc}^{\backepsilon}\, \mathsf{p} & (37.23.\mathrm{b}) \\
 & = & \mathsf{lfp}^{\dot{\subseteq}}\, \overleftrightarrow{\mathscr{F}}^{\backepsilon}_{\Pi}, & \overleftrightarrow{\mathscr{F}}^{\backepsilon}_{\Pi}(\mathsf{p}) & \triangleq & \dot{E} \,\dot{\cup}\, \mathsf{p} \,\dot{\circledcirc}^{\backepsilon}\, \mathsf{p} & (37.23.\mathrm{c}) \\
 & = & \mathsf{lfp}^{\dot{\subseteq}}_{\dot{E}}\, \widehat{\mathscr{F}}^{\backepsilon}_{\Pi}, & \widehat{\mathscr{F}}^{\backepsilon}_{\Pi}(\mathsf{p}) & \triangleq & \mathsf{p} \,\dot{\cup}\, \mathsf{p} \,\dot{\circledcirc}^{\backepsilon}\, \mathsf{p} & (37.23.\mathrm{d})
\end{array}
$$

where $\mathsf{p}_1 \,\dot{\circledcirc}^{\backepsilon}\, \mathsf{p}_2 \triangleq x, y \mapsto \bigcup_{z\in V} \{\pi_1 \odot \pi_2 \mid \pi_1 \in \mathsf{p}_1(x, z) \wedge \pi_2 \in \mathsf{p}_2(z, y) \wedge \mathsf{elem\text{-}conc?}(\pi_1, \pi_2)\}$.

$\square$

Proof of Theorem 37.23 We apply Theorem 37.6 with abstraction $\dot{\alpha}^{\backepsilon} \circ \alpha^{\circ\!-\!\circ}$ so that we have to abstract the transformers in Theorem 37.10 using an exact fixpoint abstraction of Theorem 16.17. The commutation condition yields the transformers by calculational design.

$$
\begin{array}{ll}
 & \dot{\alpha}^{\backepsilon}(\mathsf{p}_1 \,\overset{\dot{\circ\circ}}{\circledcirc}\, \mathsf{p}_2) \\
= & \dot{\alpha}^{\backepsilon}(x, y \mapsto \bigcup_{z\in V} \mathsf{p}_1(x, z) \circledcirc \mathsf{p}_2(z, y)) \hfill \text{(def. } \overset{\dot{\circ\circ}}{\circledcirc} \text{ in Theorem 37.10)} \\
= & x, y \mapsto \alpha^{\backepsilon}(\bigcup_{z\in V} \mathsf{p}_1(x, z) \circledcirc \mathsf{p}_2(z, y)) \hfill \text{(pointwise def. } \dot{\alpha}^{\backepsilon}\text{)} \\
= & x, y \mapsto \bigcup_{z\in V} \alpha^{\backepsilon}(\mathsf{p}_1(x, z) \circledcirc \mathsf{p}_2(z, y)) \hfill \text{(join preservation of the abstraction in a Galois connection)} \\
= & x, y \mapsto \bigcup_{z\in V} \alpha^{\backepsilon}(\{\pi_1 \odot \pi_2 \mid \pi_1 \in \mathsf{p}_1(x, z) \wedge \pi_2 \in \mathsf{p}_2(z, y)\}) \hfill \text{(def. (37.3) of } \circledcirc \text{ and (37.9) of } \odot\text{)} \\
= & x, y \mapsto \bigcup_{z\in V} \alpha^{\backepsilon}(\{\pi_1 \odot \pi_2 \mid \pi_1 \in \alpha^{\backepsilon}(\mathsf{p}_1(x, z)) \wedge \pi_2 \in \alpha^{\backepsilon}(\mathsf{p}_2(z, y))\}) \\
 & \hfill \text{(since if } \pi_1 \text{ or } \pi_2 \text{ are not elementary so is their concatenation } \pi_1 \odot \pi_2\text{)} \\
= & x, y \mapsto \bigcup_{z\in V} \{\pi_1 \odot \pi_2 \mid \pi_1 \in \alpha^{\backepsilon}(\mathsf{p}_1(x, z)) \wedge \pi_2 \in \alpha^{\backepsilon}(\mathsf{p}_2(z, y)) \wedge \mathsf{elem\text{-}conc?}(\pi_1, \pi_2)\} \\
 & \text{(since, by Lemma 37.21, } \pi_1 \text{ and } \pi_2 \text{ being elementary, their concatenation } \pi_1 \odot \pi_2 \text{ is elementary if and only if } \mathsf{elem\text{-}conc?}(\pi_1, \pi_2) \text{ is true)} \\
= & x, y \mapsto \bigcup_{z\in V} \{\pi_1 \odot \pi_2 \mid \pi_1 \in \dot{\alpha}^{\backepsilon}(\mathsf{p}_1)(x, z) \wedge \pi_2 \in \dot{\alpha}^{\backepsilon}(\mathsf{p}_2)(z, y) \wedge \mathsf{elem\text{-}conc?}(\pi_1, \pi_2)\} \\
 & \hfill \text{(pointwise def. } \dot{\alpha}^{\backepsilon}\text{)} \\
= & \dot{\alpha}^{\backepsilon}(\mathsf{p}_1) \,\dot{\circledcirc}^{\backepsilon}\, \dot{\alpha}^{\backepsilon}(\mathsf{p}_2) \hfill \text{(def. } \dot{\circledcirc}^{\backepsilon} \text{ in Theorem 37.23)}
\end{array}
$$

$\square$

37.14 Calculational design of the elementary paths between vertices of finite graphs

In finite graphs $G = \langle V, E\rangle$ with $|V| = n > 0$ vertices, elementary paths in are of length at most $n+1$ (for a cycle that would go through all vertices of the graph). This ensures that the fixpoint iterates in Theorem 37.23 starting from $\dot{\varnothing}$ do converge in at most $n+2$ iterates.

Moreover, if $V = \{z_1 \ldots z_n\}$ is finite, then the elementary paths of the $k+2^{\text{nd}}$ iterate can be restricted to $\{z_1, \ldots, z_k\}$. This yields an iterative algorithm by application of the exact iterates multi-abstraction Theorem 16.27 with[46]

$$\begin{array}{lcll}
\alpha_0^{\ni}(\mathsf{p}) & \triangleq & \mathsf{p} & \\
\alpha_k^{\ni}(\mathsf{p}) & \triangleq & x, y \mapsto \{\pi \in \mathsf{p}(x, y) \mid \mathsf{V}(\pi) \subseteq \{z_1, \ldots, z_k\} \cup \{x, y\}\}, & k \in [1, n] \\
\alpha_k^{\ni}(\mathsf{p}) & \triangleq & \mathsf{p}, & k > n
\end{array} \tag{37.24}$$

By the exclusion abstraction of Exercise 11.2 and pointwise extension of Exercise 11.17, these are Galois connections

$$\langle V \times V \to \wp(V^{>1}),\ \dot{\subseteq}\rangle \xleftarrow[\alpha_k^{\ni}]{\gamma_k^{\ni}} \langle V \times V \to \bigcup_{k=2}^{n+1} V^k,\ \dot{\subseteq}\rangle. \tag{37.25}$$

Applying Theorem 16.27, we get the following iterative characterization of the elementary paths of a finite graph. Notice that $\dot{\circledcirc}_z$ in (37.26.a) and (37.26.b) does not require to test that the concatenation of two elementary paths is elementary while $\dot{\circledcirc}_z^{\ni}$ in (37.26.c) and (37.26.d) definitely does (since the concatenated elementary paths may have vertices in common). Notice also that the iteration $\langle \overrightarrow{\mathscr{F}}_{\pi}^{\ni k}$, $k \in [0, n+2]\rangle$ in (37.26.a) is not the same as the iterates $\langle \overrightarrow{\mathscr{F}}_{\Pi}^{\ni k}(\dot{\varnothing}),\ k \in \mathbb{N}\rangle$ of $\overrightarrow{\mathscr{F}}_{\Pi}^{\ni}$ from $\dot{\varnothing}$, since using $\dot{\circledcirc}_z$ or $\dot{\circledcirc}_z^{\ni}$ instead of $\dot{\circledcirc}^{\ni}$ is the key to efficiency. This is also the case for (37.26.b—d).

Theorem 37.26 (Iterative characterization of the elementary paths of a finite graph) Let $G = \langle V, E\rangle$ be a finite graph with $V = \{z_1, \ldots, z_n\}$, $n > 0$. Then

$$\mathsf{p}^{\ni} = \mathsf{lfp}^{\dot{\subseteq}}\, \overrightarrow{\mathscr{F}}_{\Pi}^{\ni} = \overrightarrow{\mathscr{F}}_{\pi}^{\ni n+2} \tag{37.26.a}$$

$$\text{where}\quad \overrightarrow{\mathscr{F}}_{\pi}^{\ni 0} \triangleq \dot{\varnothing},\quad \overrightarrow{\mathscr{F}}_{\pi}^{\ni 1} \triangleq \dot{E},\quad \overrightarrow{\mathscr{F}}_{\pi}^{\ni k+2} \triangleq \dot{E} \mathbin{\dot{\cup}} \overrightarrow{\mathscr{F}}_{\pi}^{\ni k+1} \mathbin{\dot{\circledcirc}_{z_{k+1}}} \dot{E},\quad k \in [0, n]$$

$$\overrightarrow{\mathscr{F}}_{\pi}^{\ni k+1} = \overrightarrow{\mathscr{F}}_{\pi}^{\ni k},\quad k \geqslant n+2$$

$$= \mathsf{lfp}^{\dot{\subseteq}}\, \overleftarrow{\mathscr{F}}_{\Pi}^{\ni} = \overleftarrow{\mathscr{F}}_{\Pi}^{\ni n+2} \tag{37.26.b}$$

$$\text{where}\quad \overleftarrow{\mathscr{F}}_{\pi}^{\ni 0} \triangleq \dot{\varnothing},\quad \overleftarrow{\mathscr{F}}_{\pi}^{\ni 0} \triangleq \dot{E},\quad \overleftarrow{\mathscr{F}}_{\pi}^{\ni k+2} \triangleq \dot{E} \mathbin{\dot{\cup}} \dot{E} \mathbin{\dot{\circledcirc}_{z_{k+1}}} \overleftarrow{\mathscr{F}}_{\pi}^{\ni k+1},\quad k \in [0, n]$$

$$\overleftarrow{\mathscr{F}}_{\pi}^{\ni k+1} = \overleftarrow{\mathscr{F}}_{\pi}^{\ni k},\quad k \geqslant n+2$$

[46] This is for case (37.26.d). For cases (a–c), we also have $\alpha_1^{\ni}(\mathsf{p}) \triangleq \mathsf{p}$ while the second definition is for $k \in [2, n+2]$.

$$= \mathsf{lfp}^{\dot\subseteq}\, \overrightarrow{\mathscr{F}}^{\partial}_{\Pi} = \overrightarrow{\mathscr{F}}^{\partial\, n+2}_{\pi} \tag{37.26.c}$$
$$\text{where} \quad \overrightarrow{\mathscr{F}}^{\partial\, 0}_{\pi} \triangleq \dot\varnothing, \quad \overrightarrow{\mathscr{F}}^{\partial\, 1}_{\pi} \triangleq \dot E, \quad \overrightarrow{\mathscr{F}}^{\partial\, k+2}_{\pi} \triangleq \dot E \mathbin{\dot\cup} \overrightarrow{\mathscr{F}}^{\partial\, k+1}_{\pi} \mathbin{\dot\circledcirc^{\partial}_{z_{k+1}}} \overrightarrow{\mathscr{F}}^{\partial\, k+1}_{\pi}, \quad k \in [0, n]$$
$$\overrightarrow{\mathscr{F}}^{\partial\, k+1}_{\pi} = \overrightarrow{\mathscr{F}}^{\partial\, k}_{\pi}, \quad k \geqslant n+2$$

$$= \mathsf{lfp}^{\dot\subseteq}_{\dot E}\, \widehat{\mathscr{F}}^{\partial}_{\Pi} = \widehat{\mathscr{F}}^{\partial\, n+1}_{\pi} \tag{37.26.d}$$
$$\text{where} \quad \widehat{\mathscr{F}}^{\partial\, 0}_{\pi} \triangleq \dot E, \quad \widehat{\mathscr{F}}^{\partial\, k+1}_{\pi} \triangleq \widehat{\mathscr{F}}^{\partial\, k}_{\pi} \mathbin{\dot\cup} \widehat{\mathscr{F}}^{\partial\, k}_{\pi} \mathbin{\dot\circledcirc^{\partial}_{z_{k+1}}} \widehat{\mathscr{F}}^{\partial\, k}_{\pi}, \quad k \in [0, n]$$
$$\widehat{\mathscr{F}}^{\partial\, k+1}_{\pi} = \widehat{\mathscr{F}}^{\partial\, k}_{\pi}, \quad k \geqslant n+2$$

$\mathsf{p}_1 \mathbin{\dot\circledcirc_z} \mathsf{p}_2 \triangleq x, y \mapsto \{\pi_1 \odot \pi_2 \mid \pi_1 \in \mathsf{p}_1(x, z) \wedge \pi_2 \in \mathsf{p}_2(z, y) \wedge z \notin \{x, y\}\}$, and $\mathsf{p}_1 \mathbin{\dot\circledcirc^{\partial}_z} \mathsf{p}_2 \triangleq x, y \mapsto \{\pi_1 \odot \pi_2 \mid \pi_1 \in \mathsf{p}_1(x, z) \wedge \pi_2 \in \mathsf{p}_2(z, y) \wedge \mathsf{elem\text{-}conc?}(\pi_1, \pi_2)\}$. □

Proof of Theorem 37.26 — The proofs in cases (37.26.c) and (37.26.d) are similar. Let us consider (37.26.d). Assume $V = \{z_1 \dots z_n\}$ and let $\widehat{\mathscr{F}}^{\partial\, k+1}_{\Pi} = \widehat{\mathscr{F}}^{\partial}_{\Pi}(\widehat{\mathscr{F}}^{\partial\, k}_{\Pi})$ be the iterates of $\widehat{\mathscr{F}}^{\partial}_{\Pi}$ from $\widehat{\mathscr{F}}^{\partial\, 0}_{\Pi} = \dot E$ in (37.23.d). To apply Theorem 16.27, we consider the concrete cpo $\langle \mathcal{C}, \dot\subseteq\rangle$ and the abstract cpo $\langle \mathcal{A}, \dot\subseteq\rangle$ to be $\langle \mathcal{C}, \dot\subseteq, \dot E, \dot\bigcup\rangle$ with $\mathcal{C} \triangleq x \in V \times y \in V \to \{x\pi y \mid x\pi y \in E \cup \bigcup_{k=3}^{n+1} V^k \wedge x\pi y \text{ is elementary}\}$, and the functions $\widehat{\mathscr{F}}^{\partial}_{\pi k}(X) \triangleq X \mathbin{\dot\cup} X \mathbin{\dot\circledcirc^{\partial}_{z_{k+1}}} X$, $k \in [1, n]$, and $\widehat{\mathscr{F}}^{\partial}_{\pi k}(X) \triangleq X$, $k = 0$ or $k > n$ which iterates from the infimum $\dot E$ are precisely $\langle \widehat{\mathscr{F}}^{\partial\, k}_{\pi}, i \in \mathbb{Z} \cup \{\omega\}\rangle$ where $\widehat{\mathscr{F}}^{\partial\, \omega}_{\pi} = \dot{\bigcup\limits_{i \in \mathbb{Z}}} \widehat{\mathscr{F}}^{\partial\, i}_{\pi} = \widehat{\mathscr{F}}^{\partial\, n+1}_{\pi} = \widehat{\mathscr{F}}^{\partial\, k}_{\pi}$, $k > n$.

- For the infimum $\widehat{\mathscr{F}}^{\partial}_{\pi 0} = \dot E$ which paths $x\pi y \in \dot E(x, y)$ are elementary and have all intermediate states of π in $\varnothing = \{z_1, \dots, z_0\}$ since π is empty.

- For the commutation, the case $k > n$ is trivial. Otherwise let $X \in \mathcal{A}$ so $x\pi y \in X(x, y)$ is elementary and has all states of π in $\{z_1, \dots, z_k\}$

$$\alpha^{\partial}_{k+1}(\widehat{\mathscr{F}}^{\partial}_{\Pi}(X))$$
$$= \alpha^{\partial}_{k+1}(X \mathbin{\dot\cup} X \mathbin{\dot\circledcirc^{\partial}} X) \qquad \wr \text{def. (37.23.d) of } \widehat{\mathscr{F}}^{\partial}_{\Pi} \wr$$
$$= \alpha^{\partial}_{k+1}(X) \mathbin{\dot\cup} \alpha^{\partial}_{k+1}(X \mathbin{\dot\circledcirc^{\partial}} X) \qquad \wr \alpha^{\partial}_{k+1} \text{ preserves joins in (37.25)} \wr$$
$$= \alpha^{\partial}_{k}(X) \mathbin{\dot\cup} \alpha^{\partial}_{k+1}(X \mathbin{\dot\circledcirc^{\partial}} X)$$
$\wr$def. (37.24) of α^{∂}_{k+1} and hypothesis that all paths in X have all intermediate states in $\{z_1, \dots, z_k\}\wr$
$$= \alpha^{\partial}_{k}(X) \mathbin{\dot\cup} x, y \mapsto \{\pi \in X \mathbin{\dot\circledcirc^{\partial}} X(x, y) \mid \mathsf{V}(\pi) \subseteq \{z_1, \dots, z_{k+1}\} \cup \{x, y\}\} \qquad \wr \text{def. } \alpha^{\partial}_{k+1} \text{ in (37.24)} \wr$$
$$= \alpha^{\partial}_{k}(X) \mathbin{\dot\cup} x, y \mapsto \{\pi \in \bigcup_{z \in V} \{\pi_1 \odot \pi_2 \mid \pi_1 \in X(x, z) \wedge \pi_2 \in X(z, y) \wedge \mathsf{elem\text{-}conc?}(\pi_1, \pi_2)\} \mid \mathsf{V}(\pi) \subseteq \{z_1, \dots, z_{k+1}\} \cup \{x, y\}\} \qquad \wr \text{def. } \dot\circledcirc^{\partial} \text{ in Theorem 37.23} \wr$$
$$= \alpha^{\partial}_{k}(X) \mathbin{\dot\cup} x, y \mapsto \bigcup_{z \in V} \{\pi_1 \odot \pi_2 \mid \pi_1 \in X(x, z) \wedge \pi_2 \in X(z, y) \wedge \mathsf{elem\text{-}conc?}(\pi_1, \pi_2) \wedge \mathsf{V}(\pi_1 \odot \pi_2) \subseteq \{z_1, \dots, z_{k+1}\} \cup \{x, y\}\} \qquad \wr \text{def. } \in \wr$$

$= \alpha^{\partial}_{k}(X) \mathbin{\dot\cup} x, y \mapsto \bigcup_{z \in V} \{x\pi_1 z\pi_2 y \mid x\pi_1 z \in X(x,z) \wedge z\pi_2 y \in X(z,y) \wedge \mathsf{elem\text{-}conc?}(x\pi_1 z, z\pi_2 y) \wedge$
$\mathsf{V}(\pi_1) \cup \mathsf{V}(\pi_2) \cup \{z\} \subseteq \{z_1, \ldots, z_{k+1}\} \cup \{x, y\}\}$ $\wr$ def. $\odot$, V, and ind. hyp. $\wr$

$= \alpha^{\partial}_{k}(X) \mathbin{\dot\cup} x, y \mapsto \{x\pi_1 z_{k+1}\pi_2 y \mid x\pi_1 z_{k+1} \in \alpha^{\partial}_{k}(X)(x, z_{k+1}) \wedge z_{k+1}\pi_2 y \in \alpha^{\partial}_{k}(X)(z_{k+1}, y) \wedge$
$\mathsf{elem\text{-}conc?}(x\pi_1 z_{k+1}, z_{k+1}\pi_2 y)\}$

$\wr$ ($\supseteq$) follows from taking $z = z_{k+1}$; (37.27)
($\subseteq$) For $z \in \{z_1, \ldots, z_k\}$, the paths in $\alpha^{\partial}_{k}(X)$ are elementary through $\{z_1, \ldots, z_k\}$, so if there exist paths $x\pi_1 z \in X(x,z)$ and $z\pi_2 y \in X(z,y)$ then either $x\pi_1 z\pi_2 x$ is also elementary through $\{z_1, \ldots, z_k\}$ and already therefore belongs to $\alpha^{\partial}_{k}(X)$ or it is not elementary and then does not pass the test $\mathsf{elem\text{-}conc?}(x\pi_1 z_{k+1}, z_{k+1}\pi_2 y)$;
Otherwise, if $z \in \{z_{k+2}, \ldots, z_n\}$, then the path $x\pi_1 z_{k+1}\pi_2 y$ is eliminated by $\mathsf{V}(\pi_1) \cup \mathsf{V}(\pi_2) \cup \{z\} \subseteq \{z_1, \ldots, z_{k+1}\} \cup \{x, y\}$;
Finally, the only possibility is $z = z_{k+1}$, in which case all paths have the form $x\pi_1 z_{k+1}\pi_2 y$, are elementary, and with $\mathsf{V}(\pi) \subseteq \{z_1, \ldots, z_{k+1}\}$, as required by the def. of $\mathcal{A}$. It also holds for $\alpha^{\partial}_{k}(X)$ which is equal to $\alpha^{\partial}_{k+1}(X)$. $\wr$

$= \alpha^{\partial}_{k}(X) \mathbin{\dot\cup} x, y \mapsto \{x\pi_1 z_{k+1} \odot z_{k+1}\pi_2 y \mid x\pi_1 z_{k+1} \in \alpha^{\partial}_{k}(X)(x, z_{k+1}) \wedge z_{k+1}\pi_2 y \in \alpha^{\partial}_{k}(X)(z_{k+1}, y) \wedge$
$\mathsf{elem\text{-}conc?}(x\pi_1 z_{k+1}, z_{k+1}\pi_2 y)\}$ $\wr$ def. $\odot$ $\wr$

$= \alpha^{\partial}_{k}(X) \mathbin{\dot\cup} x, y \mapsto \{\pi_1 \odot \pi_2 \mid \pi_1 \in \alpha^{\partial}_{k}(X)(x, z_{k+1}) \wedge \pi_2 \in \alpha^{\partial}_{k}(X)(z_{k+1}, y) \wedge \mathsf{elem\text{-}conc?}(\pi_1, \pi_2)\}$
$\wr$ by ind. hyp. all paths in $X(x, y)$ have the form $x\pi y$ $\wr$

$= \alpha^{\partial}_{k}(X) \mathbin{\dot\cup} \alpha^{\partial}_{k}(X) \mathbin{\dot{\circledcirc}^{\partial}_{z_{k+1}}} \alpha^{\partial}_{k}(X)$ $\wr$ def. $\dot{\circledcirc}^{\partial}_{z_{k+1}}$ in (37.26.d) $\wr$

$= \widehat{\mathscr{F}}^{\partial}_{\pi k}(\alpha^{\partial}_{k}(X))$ $\wr$ (37.26.d) $\wr$

We conclude by Corollary 16.28.

— In cases (37.26.a) and (37.26.b), $\dot{\circledcirc}^{\partial}_{z_{k+1}}$ can be replaced by $\dot{\circledcirc}_{z_{k+1}}$ since in this cases the paths are elementary by construction. To see this, observe that for (37.26.a), the iterates $\langle \overrightarrow{\mathscr{F}}^{\partial\,k}_{\pi}(\dot\varnothing),\ k \in \mathbb{N} \cup \{\omega\}\rangle$ are those of the functions $\overrightarrow{\mathscr{F}}^{\partial}_{\pi 0}(X) \triangleq \dot\varnothing$, $\overrightarrow{\mathscr{F}}^{\partial}_{\pi 1}(X) \triangleq \dot{E}$, and $\overrightarrow{\mathscr{F}}^{\partial}_{\pi k}(X) \triangleq \dot{E} \mathbin{\dot\cup} X \mathbin{\dot{\circledcirc}^{\partial}_{z_{k-1}}} \dot{E}$, $k \in [2, n+2]$, and $\overrightarrow{\mathscr{F}}^{\partial}_{\pi k}(X) \triangleq X$, $k > n$, so that we can consider the iterates from 1 to apply Theorem 16.27.

- By (37.23.a), the initialization is $\overrightarrow{\mathscr{F}}^{\partial}_{\Pi}(\dot\varnothing) \triangleq \dot{E} \mathbin{\dot\cup} \dot\varnothing \mathbin{\dot{\circledcirc}^{\partial}} \dot{E} = \dot{E}$ such that all paths $x\pi y$ in $\dot{E}(x, y)$ are elementary with π empty so $\mathsf{V}(\pi) \subseteq \varnothing = \{z_1, \ldots, z_0\}$.
- For the commutation, let $X \in \mathcal{A}$ such that all $x\pi y \in X(x, y)$ are elementary and have all states of π in $\{z_1, \ldots, z_k\}$. Then

$\alpha^{\partial}_{k+2}(\overrightarrow{\mathscr{F}}^{\partial}_{\Pi}(X))$ $\wr$ def. iterates $\wr$

$= \alpha^{\partial}_{k+2}(\dot{E} \mathbin{\dot\cup} X \mathbin{\dot{\circledcirc}^{\partial}} \dot{E})$ $\wr$ def. (37.23.a) of $\overrightarrow{\mathscr{F}}^{\partial}_{\Pi}$ $\wr$

$= \alpha^{\partial}_{k+2}(\dot{E}) \mathbin{\dot\cup} \alpha^{\partial}_{k+2}(X \mathbin{\dot{\circledcirc}^{\partial}} \dot{E})$ $\wr$ α^{∂}_{k+2} preserves joins in (37.25) $\wr$

$= \dot{E} \,\dot{\cup}\, x, y \mapsto \{\pi \in X \mathbin{\dot{\circledcirc}^{\partial}} \dot{E} \mid \mathsf{V}(\pi) \subseteq \{z_1, \ldots, z_{k+2}\} \cup \{x, y\}\}$ ⟅def. α^{∂}_{k+2} in (37.24)⟆

$= \dot{E} \,\dot{\cup}\, x, y \mapsto \{\pi \in \bigcup_{z \in V} \{\pi_1 \odot \pi_2 \mid \pi_1 \in X(x, z) \wedge \pi_2 \in \dot{E}(z, y) \wedge \textsf{elem-conc?}(\pi_1, \pi_2)\} \mid \mathsf{V}(\pi) \subseteq \{z_1, \ldots, z_{k+2}\} \cup \{x, y\}\}$ ⟅def. $\dot{\circledcirc}^{\partial}$ in Theorem 37.23⟆

$= \dot{E} \,\dot{\cup}\, x, y \mapsto \bigcup_{z \in V} \{\pi_1 \odot \pi_2 \mid \pi_1 \in X(x, z) \wedge \pi_2 \in \dot{E}(z, y) \wedge \textsf{elem-conc?}(\pi_1, \pi_2) \wedge \mathsf{V}(\pi_1 \odot \pi_2) \subseteq \{z_1, \ldots, z_{k+2}\} \cup \{x, y\}\}$ ⟅def. $\in$⟆

$= \dot{E} \,\dot{\cup}\, x, y \mapsto \bigcup_{z \in V} \{x\pi_1 z y \mid x\pi_1 z \in X(x, z) \wedge z y \in \dot{E}(z, y) \wedge \textsf{elem-conc?}(x\pi_1 z, z y) \wedge \mathsf{V}(\pi_1) \cup \{z\} \subseteq \{z_1, \ldots, z_{k+2}\} \cup \{x, y\}\}$ ⟅def. $\odot$, V, $\dot{E}$ in Theorem 37.10, and ind. hyp.⟆

$= \dot{E} \,\dot{\cup}\, x, y \mapsto \{x\pi_1 z_{k+2} \odot z_{k+2}\pi_2 y \mid x\pi_1 z_{k+2} \in \alpha^{\partial}_{k+1}(X)(x, z_{k+2}) \wedge z_{k+2}\pi_2 y \in \dot{E}(z_{k+2}, y) \wedge \textsf{elem-conc?}(x\pi_1 z_{k+2}, z_{k+2}\pi_2 y)\}$ ⟅by an argument similar to (37.27)⟆

$= \dot{E} \,\dot{\cup}\, x, y \mapsto \{x\pi_1 z_{k+2} \odot z_{k+2} y \mid x\pi_1 z_{k+2} \in \alpha^{\partial}_{k+1}(X)(x, z_{k+2}) \wedge \langle z_{k+2},\ y\rangle \in E \wedge \textsf{elem-conc?}(x\pi_1 z_{k+2}, z_{k+2} y)\}$ ⟅def. (37.24) of α^{∂}_{k+1} and $\dot{E}$ in Theorem 37.10⟆

$= \dot{E} \,\dot{\cup}\, x, y \mapsto \{x\pi_1 z_{k+2} \odot z_{k+2} y \mid x\pi_1 z_{k+2} \in \alpha^{\partial}_{k+1}(X)(x, z_{k+2}) \wedge \langle z_{k+2},\ y\rangle \in E\}$
⟅since $z_{k+2} \notin \mathsf{V}(\pi_1)$ by induction hypothesis path so that the path $x\pi_1 z_{k+2} y$ is elementary⟆

$= \dot{E} \,\dot{\cup}\, \alpha^{\partial}_{k+1}(X) \mathbin{\dot{\circledcirc}_{z_{k+2}}} \dot{E}$ ⟅def. $\dot{\circledcirc}^{\partial}_{z_{k+2}}$ in Theorem 37.26⟆

$= \vec{\mathscr{F}}^{\partial}_{\pi k+2}(\alpha^{\partial}_{k+1}(X))$ ⟅(37.26.a)⟆ □

37.15 Calculational design of an over-approximation of the elementary paths between vertices of finite graphs

Since shortest paths are necessarily elementary, one could expect that Roy-Floyd-Warshall algorithm simply abstracts the elementary paths by their length. This is not the case, because the iterations in (37.26.c) and (37.26.d) for elementary paths are too expensive. They require to check elem-conc? in $\dot{\circledcirc}^{\partial}$ to make sure that the concatenation of elementary paths does yield an elementary path. But we can over-approximate by replacing $\dot{\circledcirc}^{\partial}$ by $\dot{\circledcirc}$ since the length of the shortest paths in the graph is the same as the length of the shortest paths in any subset of the graph paths provided this subset contains all elementary paths.

Corollary 37.28 (Iterative characterization of an over-approximation of the elementary paths of a finite graph) Let $G = \langle V,\ E\rangle$ be a finite graph with $V = \{z_1, \ldots, z_n\}$, $n > 0$. Then

$$\begin{aligned}
\mathsf{p}^{\partial} &= \mathsf{lfp}^{\dot{\subseteq}}\ \overleftrightarrow{\mathscr{F}}^{\partial}_{\Pi} \ \dot{\subseteq}\ \overleftrightarrow{\mathscr{F}}^{n+2}_{\boldsymbol{\pi}} && (37.28.\text{c})\\
&\qquad \text{where}\quad \overleftrightarrow{\mathscr{F}}^{0}_{\boldsymbol{\pi}} \triangleq \dot{\varnothing},\quad \overleftrightarrow{\mathscr{F}}^{1}_{\boldsymbol{\pi}} \triangleq \dot{E},\quad \overleftrightarrow{\mathscr{F}}^{k+2}_{\boldsymbol{\pi}} \triangleq \dot{E} \mathbin{\dot{\cup}} \overleftrightarrow{\mathscr{F}}^{k+1}_{\boldsymbol{\pi}} \mathbin{\dot{\circledcirc}_{z_{k+1}}} \overleftrightarrow{\mathscr{F}}^{k+1}_{\boldsymbol{\pi}}\\
&= \mathsf{lfp}^{\dot{\subseteq}}_{\dot{E}}\ \widehat{\mathscr{F}}^{\partial}_{\Pi} \ \dot{\subseteq}\ \widehat{\mathscr{F}}^{n+1}_{\boldsymbol{\pi}} && (37.28.\text{d})\\
&\qquad \text{where}\quad \widehat{\mathscr{F}}^{0}_{\boldsymbol{\pi}} \triangleq \dot{E},\quad \widehat{\mathscr{F}}^{k+1}_{\boldsymbol{\pi}} \triangleq \widehat{\mathscr{F}}^{k}_{\boldsymbol{\pi}} \mathbin{\dot{\cup}} \widehat{\mathscr{F}}^{k}_{\boldsymbol{\pi}} \mathbin{\dot{\circledcirc}_{z_k}} \widehat{\mathscr{F}}^{k}_{\boldsymbol{\pi}} && \square
\end{aligned}$$

Proof of Corollary 37.28 Obviously $\dot{\circledcirc}^{\partial}_{z} \ \dot{\subseteq}\ \dot{\circledcirc}_{z}$ so the iterates $\langle \overleftrightarrow{\mathscr{F}}^{k}_{\boldsymbol{\pi}},\ k \in [0, n+2]\rangle$ of (37.28.c) over-approximate those $\langle \overleftrightarrow{\mathscr{F}}^{\partial k}_{\pi},\ k \in [0, n+2]\rangle$ of (37.26.c). Same for (37.26.d). $\square$

Exercise 37.29. Show that in (37.28.d), the subset inclusion can be strict. $\square$

37.16 The Roy-Floyd-Warshall algorithm over-approximating the elementary paths of a finite graph

The Roy-Floyd-Warshall algorithm does <u>not</u> computes elementary paths in (37.26.d) but the over-approximation of the set of elementary paths in (37.28.d), thus avoiding the potentially costly test in Theorem 37.26 that the concatenation of elementary paths in $\dot{\circledcirc}^{\partial}_{z}$ is elementary.

Proof of Algorithm ?? The first for iteration computes $\widehat{\mathscr{F}}^{0}_{\boldsymbol{\pi}} \triangleq \dot{E}$ in (37.28.d). Then, the second for iteration should compute $\widehat{\mathscr{F}}^{k+1}_{\boldsymbol{\pi}} \triangleq \widehat{\mathscr{F}}^{k}_{\boldsymbol{\pi}} \mathbin{\dot{\cup}} \widehat{\mathscr{F}}^{k}_{\boldsymbol{\pi}} \mathbin{\dot{\circledcirc}_{z_k}} \widehat{\mathscr{F}}^{k}_{\boldsymbol{\pi}}$ in (37.28.d) since $\mathsf{p}_1 \mathbin{\dot{\circledcirc}_{z}} \mathsf{p}_2 = \dot{\varnothing}$ in (37.26.d) when $z \in \{x, y\}$, in which case, $\widehat{\mathscr{F}}^{k+1}_{\boldsymbol{\pi}} = \widehat{\mathscr{F}}^{k}_{\boldsymbol{\pi}}$, which is similar to the Jacobi iterative method in Section **20.2**. However, similar to the Gauss-Seidel iteration method in Section **20.2**, we reuse the last computed $\mathsf{p}(x, z)$ and $\mathsf{p}(z, y)$, not necessarily those of the previous iterate. For the convergence of the first n iterates of the second for iteration of the algorithm, the justification is similar to the convergence Corollary 20.6 for chaotic iterations in Definition 20.2. $\square$

37.17 Calculational design of the Roy-Floyd-Warshall shortest path algorithm

The shortest path algorithm of Bernard Roy [14], Bob Floyd [8], and Steve Warshall [18] for finite graphs is based on the assumption that the graph has no cycles with strictly negative weights *i.e.* $\forall x \in V\ .\ \forall \pi \in \mathsf{p}(x, x)\ .\ \boldsymbol{\omega}(\pi) \geqslant 0$. In that case the shortest paths are all elementary since adding a cycle of weight 0 leaves the distance unchanged while a cycle of positive weight would strictly increase the distance on the path. Otherwise, as shown by Example 37.16, if the graph has cycle with strictly negative weights, the convergence between two vertices containing a cycle with strictly negative weights is infinite to the limit $-\infty$.

The essential consequence is that we don't have to consider all paths as in Theorem 37.10 but instead we can consider any subset provided that it contains all elementary paths. Therefore we can base the design of the shortest path algorithm on Corollary 37.28. Observe that, although p may contain paths that are not elementary, d is precisely the minimal path lengths and not some strict over-approximation since

- p contains all elementary paths (so non-elementary paths are longer than the elementary path between their extremities), and
- no arc has a strictly negative weight (so path lengths are always positive and therefore the elementary paths are the shortest ones).

We derive the Roy-Floyd-Warshall algorithm by a calculation design applying Exercise **??** for finite iterates to (37.28.d) with the abstraction $\dot{\boldsymbol{\omega}}$ (or a variant when considering (37.28.c)).

– for the infimum $\dot{E}$ in (37.28.d), we have

$$\begin{aligned}
&\dot{\boldsymbol{\omega}}(\dot{E})\langle x,\ y\rangle \\
=\ & \boldsymbol{\omega}(\dot{E}(x, y)) && \wr\text{pointwise def. } \dot{\boldsymbol{\omega}}\int \\
=\ & \boldsymbol{\omega}(\llbracket \langle x,\ y\rangle \in E \mathrel{?} \{\langle x,\ y\rangle\} \mathrel{:} \varnothing \rrbracket) && \wr\text{def. } \dot{E} \text{ in Theorem 37.10}\int \\
=\ & \llbracket \langle x,\ y\rangle \in E \mathrel{?} \boldsymbol{\omega}(\{\langle x,\ y\rangle\}) \mathrel{:} \boldsymbol{\omega}(\varnothing) \rrbracket && \wr\text{def. conditional}\int \\
=\ & \llbracket \langle x,\ y\rangle \in E \mathrel{?} \min\{\boldsymbol{\omega}(\pi) \mid \pi \in \{\langle x,\ y\rangle\}\} \mathrel{:} \infty \rrbracket && \wr(37.14)\int \\
=\ & \llbracket \langle x,\ y\rangle \in E \mathrel{?} \boldsymbol{\omega}(x, y) \mathrel{:} \infty \rrbracket && \wr(37.13)\int
\end{aligned}$$

– for the commutation with $\widehat{\mathscr{F}}_{\boldsymbol{\pi}k+1}(X) \quad \triangleq \quad X \mathbin{\dot{\cup}} X \mathbin{\dot{\circledcirc}_{z_k}} X$, we have

$$\begin{aligned}
&\dot{\boldsymbol{\omega}}(\widehat{\mathscr{F}}_{\boldsymbol{\pi}k+1}(X))\langle x,\ y\rangle \\
=\ & \dot{\boldsymbol{\omega}}(X \mathbin{\dot{\cup}} X \mathbin{\dot{\circledcirc}_{z_k}} X)\langle x,\ y\rangle && \wr(37.28.\text{d})\int \\
=\ & \min(\dot{\boldsymbol{\omega}}(X)\langle x,\ y\rangle, \dot{\boldsymbol{\omega}}(X \mathbin{\dot{\circledcirc}_{z_k}} X)\langle x,\ y\rangle) \\
& && \wr\text{the abstraction } \dot{\boldsymbol{\omega}} \text{ of Galois connection (37.15) preserves existing joins}\int
\end{aligned}$$

Let us evaluate

$$\begin{aligned}
&\dot{\boldsymbol{\omega}}(X \mathbin{\dot{\circledcirc}_{z_k}} X)\langle x,\ y\rangle \\
=\ & \boldsymbol{\omega}((X \mathbin{\dot{\circledcirc}_{z_k}} X)(x, y)) && \wr\text{pointwise def. } \dot{\boldsymbol{\omega}}\int \\
=\ & \boldsymbol{\omega}(\{\pi_1 \odot \pi_2 \mid \pi_1 \in X(x, z_k) \wedge \pi_2 \in X(z_k, y) \wedge z_k \notin \{x, y\}\}) && \wr\text{def. } \dot{\circledcirc}_{z_k} \text{ in Theorem 37.26}\int \\
=\ & \min\{\boldsymbol{\omega}(\pi_1 \odot \pi_2) \mid \pi_1 \in X(x, z_k) \wedge \pi_2 \in X(z_k, y) \wedge z_k \notin \{x, y\}\} && \wr(37.14)\int \\
=\ & \min\{\boldsymbol{\omega}(\pi_1) + \boldsymbol{\omega}(\pi_2) \mid \pi_1 \in X(x, z_k) \wedge \pi_2 \in X(z_k, y) \wedge z_k \notin \{x, y\}\} && \wr\text{def. (37.13) of } \boldsymbol{\omega}\int
\end{aligned}$$

$= \llbracket z_k \in \{x, y\} \mathrel{?} \infty \mathrel{:} \min\{\boldsymbol{\omega}(\pi_1) \mid \pi_1 \in X(x, z_k)\} + \min\{\boldsymbol{\omega}(\pi_2) \mid \pi_1 \in X(x, z_k) \wedge \pi_2 \in X(z_k, y)\} \rrbracket$
$\wr$def. min$\wr$
$= \llbracket z_k \in \{x, y\} \mathrel{?} \infty \mathrel{:} \min(\dot{\boldsymbol{\omega}}(X)(x, z_k)) + \min(\dot{\boldsymbol{\omega}}(X)(z_k, y)) \rrbracket$ $\wr$(37.14) and pointwise def. $\dot{\boldsymbol{\omega}}\wr$
so that $\dot{\boldsymbol{\omega}}(\widehat{\mathscr{F}}_{\pi k+1}(X)) = \widehat{\mathscr{F}}_{\delta k}(\dot{\boldsymbol{\omega}}(X))$ with $\widehat{\mathscr{F}}_{\delta k}(X)(x, y) \triangleq \llbracket z_k \in \{x, y\} \mathrel{?} X(x, y) \mathrel{:} \min(X(x, y), X(x, z_k) + X(z_k, y)) \rrbracket$.

We have proved

Theorem 37.30 (Iterative characterization of the shortest path length of a graph) Let $G = \langle V, E, \boldsymbol{\omega}\rangle$ be a finite graph with $V = \{z_1, \ldots, z_n\}$, $n > 0$ weighted on the totally ordered group $\langle \mathbb{G}, \leqslant, 0, +\rangle$ with no strictly negative weight. Then the distances between any two vertices are

$$\mathsf{d} = \dot{\boldsymbol{\omega}}(\mathsf{p}) = \widehat{\mathscr{F}}_\delta^{n+1} \quad \text{where} \tag{37.30}$$
$$\widehat{\mathscr{F}}_\delta^{0}(x, y) \triangleq \llbracket \langle x, y\rangle \in E \mathrel{?} \boldsymbol{\omega}(x, y) \mathrel{:} \infty \rrbracket,$$
$$\widehat{\mathscr{F}}_\delta^{k+1}(x, y) \triangleq \llbracket z_k \in \{x, y\} \mathrel{?} \widehat{\mathscr{F}}_\delta^{k}(x, y) \mathrel{:} \min(\widehat{\mathscr{F}}_\delta^{k}(x, y), \widehat{\mathscr{F}}_\delta^{k}(x, z_k) + \widehat{\mathscr{F}}_\delta^{k}(z_k, y)) \rrbracket \quad \square$$

and directly get the Roy-Floyd-Warshall distances algorithm.

Proof of Algorithm ?? Instead of calculating the next iterate $\widehat{\mathscr{F}}_\delta^{k+1}$ as a function of the previous one $\widehat{\mathscr{F}}_\delta^{k}$ (à la Jacobi), we reuse the latest assigned values (à la Gauss-Seidel), as authorized by the chaotic iteration Corollary 20.6. $\square$

Exercise 37.31. Show that the Roy-Floyd-Warshall would be incorrect for the longest elementary path length of a graph. Which elementary path fixpoint computation would you recommend for this problem? Design the algorithm by calculus. $\square$

Exercise 37.32 (Irreflexive transitive closure). Consider the abstraction $\alpha_+(P) \triangleq \bigcup\{\langle x, y\rangle \in V^2 \mid \exists n \in \mathbb{N} \,.\, \exists z_1 \ldots z_n \in V^n \,.\, x, z_1 \ldots z_n, y \in P\}$. Define $E^+ \triangleq \alpha_+(\bigcup\{\mathsf{p}(x, y) \mid \langle x, y\rangle \in V\})$. Prove that $E^+ = \mathsf{lfp}^{\subseteq}\, X \mapsto E \cup X \mathbin{⨾} X$ using Theorem 37.10 and Theorem 16.17. $\square$

Exercise 37.33 (Reflexive transitive closure). Continuing Exercise 37.32 prove that the reflexive transitive closure is $E^* \triangleq \alpha_*(E^+) = \mathsf{lfp}^{\subseteq}\, X \mapsto \mathbb{1}_V \cup E \cup X \mathbin{⨾} X$ where $\alpha_*(R) \triangleq \mathbb{1}_V \cup R$ and $\mathbb{1}_V$ is the identity relation on V. $\square$

37.18 Adjacency matrix

The boolean adjacency matrix of a finite graph $G = \langle [1, n], E\rangle$, $n \in \mathbb{N}^+$ is $\mathbf{G} \triangleq (\llbracket \langle i, j\rangle \in E \mathrel{?} 1 \mathrel{:} 0 \rrbracket)_{\substack{i=1,n \\ j=1,n}} \in \{0, 1\}^{n\times n}$, see Example 37.1.

For a graph $G = \langle V,\ E,\ \boldsymbol{\omega}\rangle$ weighted on the group $\mathbb{G}$, the adjacency matrix is $\mathbf{G} = (\llbracket \langle i,\ j\rangle \in E \,?\, \boldsymbol{\omega}(i,j) \,:\, \infty \rrbracket)_{\substack{i=1,n\\ j=1,n}} \in (\mathbb{G}\cup\{\infty\})^{n\times n}$, see Example 37.11.

The infimum is the empty graph $\langle\varnothing,\ \varnothing\rangle$ encoded with the empty matrix $\varnothing$. The supremum is $(\infty)_{\substack{i=1,n\\ j=1,n}}$.

There is a Galois isomorphism between $\mathbf{G}$ and $\langle i,\ j\rangle \mapsto \boldsymbol{\omega}(E\cap\{\langle i,\ j\rangle\})$ and similarly the distance d can be encoded, up to an isomorphism into $\mathbf{D} = (\mathsf{d}(i,j))_{\substack{i=1,n\\ j=1,n}} \in (\mathbb{G}\cup\{\infty\})^{n\times n}$, so that by an abstraction of (37.23.c) similar to Theorem 37.17, it follows that

Corollary 37.34 (Shortest distances in a weighted graph with adjacency matrix) Let $G = \langle V,\ E,\ \boldsymbol{\omega}\rangle$ be a finite graph weighted on the totally ordered group $\langle\mathbb{G},\ \leqslant,\ 0,\ +\rangle$. Then the matrix of distances between any two vertices is $\mathbf{D} = \mathsf{lfp}^{\dot\geqslant}\, \mathscr{F}_\mu = \mathsf{gfp}^{\dot\leqslant}\, \mathscr{F}_\mu$ where $\mathscr{F}_\mu^{\mathbf{G}}(\mathbf{D})_{ij} \triangleq \min(\mathbf{G}_{ij}, \min_{k\in[1,n]}(\mathbf{D}_{ik}+\mathbf{D}_{kj}))$. □

Example 37.35 The iterates for Example 37.11 starting from $\mathbf{G}$ are $\mathscr{F}_\mu^1 = \begin{bmatrix} \infty & 1 & 2\\ -1 & \infty & 2\\ \infty & \infty & 1\end{bmatrix}$, $\mathscr{F}_\mu^2 = \begin{bmatrix} 0 & 1 & 2\\ -1 & 0 & 1\\ \infty & \infty & 1\end{bmatrix} = \mathscr{F}_\mu^3 = \mathbf{D}$. □

Example 37.36 (Graph with cycle of strictly negative weight) Continuing Example 37.16, the iterates of Corollary 37.34 starting from $\mathbf{G}$ are $\mathscr{F}_\mu^0 = \begin{bmatrix} \infty & 0 & 2\\ -1 & \infty & 2\\ \infty & \infty & 1\end{bmatrix}$, $\mathscr{F}_\mu^1 = \begin{bmatrix} -1 & 0 & 2\\ -1 & -1 & 1\\ \infty & \infty & 1\end{bmatrix}$, $\mathscr{F}_\mu^2 = \begin{bmatrix} -2 & -1 & 1\\ -2 & -2 & 0\\ \infty & \infty & 1\end{bmatrix}$, $\mathscr{F}_\mu^3 = \begin{bmatrix} -3 & -3 & -1\\ -4 & -4 & -2\\ \infty & \infty & 1\end{bmatrix}$, $\mathscr{F}_\mu^4 = \begin{bmatrix} -7 & -7 & -5\\ -8 & -8 & -6\\ \infty & \infty & 1\end{bmatrix}$, …, and passing to the limit, $\mathscr{F}_\mu^\omega = \begin{bmatrix} -\infty & -\infty & -\infty\\ -\infty & -\infty & -\infty\\ \infty & \infty & 1\end{bmatrix}$. □

An efficient iteration is obtained by abstracting (37.28.d).

Corollary 37.37 (Shortest distances in a weighted graph with adjacency matrix iteration) Let $G = \langle V,\ E,\ \boldsymbol{\omega}\rangle$ be a finite graph with $V = \{z_1, \ldots, z_n\}$, $n > 0$ weighted on the totally ordered group $\langle\mathbb{G},\ \leqslant,\ 0,\ +\rangle$ with no strictly negative weight. Then the matrix of distances between any two vertices is $\mathbf{D} = \mathscr{F}_\mu^{n+1}$ where $\mathscr{F}_\mu^0 = \mathbf{G}$ and $\mathscr{F}_{\mu\ ij}^{\ell+1} = \min(\mathscr{F}_{\mu\ ij}^{\ell}, \min_{k\in[1,n]}(\mathscr{F}_{\mu\ ik}^{\ell} + \mathscr{F}_{\mu\ kj}^{\ell}))$. □

Exercise 37.38. Continue Exercise 37.32 using the boolean adjacency matrix of the finite graph (to get the original Roy-Warshall iterative algorithm [14, 18]). □

Exercise 37.39. Program the Roy-Floyd-Warshall algorithm of Exercise 37.38 in the programming language of your choice. □

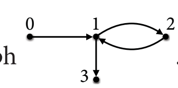
37.19 Conclusion

Graph theory originates from Leonhard Euler's work on the so-called "Seven Bridges of Königsberg" problem [7]. Graphs are used to model many types of relations and processes in physical, biological, social, and information systems.

[1, 5, 9, 12, 13] observed that various path algorithms can be designed and proved correct based on a common algebraic structure and then instantiated to various path problems up to homomorphisms. The explanation of this observation originates from the fixpoint characterization of graph paths in Theorem 37.4 based on the primitives E, $\cup$ and $\circledcirc$ on sets of paths. Then Theorem 37.6 shows that any path problem formalized by an abstraction α can be expressed in terms of abstract operations $\alpha(E)$, $\sqcup$, and $\overline{\circledcirc}$ derived from the primitives E, $\cup$ and $\circledcirc$ by calculational design. This is the case for the shortest path problem for which the composition of successive abstractions (which is not an homomorphism) allows us to formally derive the classical Roy-Floyd-Warshall algorithm (which therefore need not be postulated out of similarity observations, *e.g.* [13, Sect. 1] and [3, Ch. 3]). The derivation of Roy-Floyd-Warshall algorithm was tricky since it is based on the abstraction of an over-approximation of the elementary paths which is an under-approximation of all graph paths.

For comments on the Roy-Floyd-Warshall algorithm, see [10, p. 26–29], [11] and [16, p. 129]. The calculational design of the transitive closure by abstraction of a fixpoint path semantics is in [4]. See [17] for the calculational design of dominance and shortest graph algorithms by abstract interpretation (based on exact abstractions of (37.4.a)).

37.20 Answers to selected exercises

Answer of exercise 37.29. Consider the following graph 

. Initially $12 \in \widehat{\mathscr{F}}^{0}_{\pi}(1,2)$, $13 \in \widehat{\mathscr{F}}^{0}_{\pi}(1,3)$ and $21 \in \widehat{\mathscr{F}}^{0}_{\pi}(2,1)$. The next iterate is identical since there is no path through 0. The next iterate through $1 \notin \{2,3\}$ adds $21 \odot 13 = 213 \in \widehat{\mathscr{F}}^{2}_{\pi}(2,3)$. The next iterate through $2 \notin \{1,3\}$ adds $12 \odot 213 = 1213 \in \widehat{\mathscr{F}}^{3}_{\pi}(1,3)$ which is not elementary and so does not belong to $\mathsf{p}^{\partial}(1,3)$. □

Answer of exercise 37.39.

```
$ cat rfw.c
#include <limits.h>
#include <stdio.h>
int main () {
#define N 3
#define INF INT_MAX
    int D[N][N] = {{INF, 1, 2}, {-1, INF, 2}, {INF, INF, 1}};
```

```
    int i,j,k,dikj,negativecycle;

    for (i=0; i<N; i++) { D[i][i] = 0; }
    for (k=0; k<N; k++)
        for (i=0; i<N; i++)
            for (j=0; j<N; j++) {
                dikj = (D[i][k]==INF | D[k][j]==INF ? INF : D[i][k]+D[k][j]);
                if (dikj < D[i][j])
                    D[i][j] = dikj;
            }
    negativecycle = 0;
    for (i=0; i<N; i++) {
        if (D[i][i]<0) negativecycle = 1;
    }
    if (negativecycle) printf("cycle of strictly negative length"); else
        for (i=0; i<N; i++) {
            for (j=0; j<N; j++)
                (D[i][j]==INF ? printf("oo ") : printf("%i ", D[i][j]));
            printf ("\n");
        }
}
$ gcc rfw.c
$ ./a.out
0 1 2
-1 $0$ 1
oo oo $0$
```

□

37.21 Bibliography

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