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Hierarchies and Eingenvalue Analysis: A Generalisation to the Continuous Case.

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— ABSTRACT —

The purpose of this paper is to give an introduction to a possibly wider generalisation of the mathematical framework of the theory of hierarchies and eigenvalue analysis to the continuous case. Extensions and generalisations of expressions of eigenfunctions and basic theorems on consistency are given. A conjecture about what consistency and continuity mean is then drawn from a simple discussion about judgments.

Introduction.

The theory of hierarchies based on pairwise comparison matrices gives a means to generate, through a process of decomposition into levels of vectors of priorities, an overall priority vector whose elements (from 1 to N, say) are the final weights of the N components regarding the top objective of the hierarchy. One can then ask the question of using a similar process when the decomposition can no longer be done into a discrete set of components, but rather into a continuous interval of components. To illustrate this, let us think of an objective whose achievement depends on time — different instants of the day, say. A discrete decomposition could roughly be: [Morning, Midday, Evening, Night] for example; and the pairwise comparison analysis would give a priority vector of four components corresponding to the four moments. Now, we can also consider a day as a continuous interval of time, every point of which can be compared to any other. What was a pairwise comparison matrix in a discrete model thus becomes the set of all the values of these judgments, for any couple of instants of the day.

More generally, given an interval $[a,b]$, we can, as a first approach, assume that the judgments are defined by

$$J(s,t) \qquad (s,t) \in [a,b] \times [a,b]$$

where J is a function of the two arguments s and t, points of the "effect" interval $[a,b]$.

The properties of this function J are to translate the reciprocity of the judgments and the scaling of the judgments. This is written as:

(1) $$J(s,t) \cdot J(t,s) = 1 \qquad \forall (s,t) \in [a,b]^2$$

(2) $$0 < \frac{1}{\alpha} \leq J(s,t) \leq \alpha \qquad \forall (s,t) \in [a,b]^2$$

where α is the scale used for the judgments. (Note that for all t in $[a,b]$ $J(t,t)=1$).

Let us assume, for the moment, that the function J has all nice properties such as continuity, integrability, and so forth, with regard to both arguments, and let w be the function on $[a,b]$ which shall be the eigenfunction, solution of the integral equation:

(3) $$\int_a^b J(s,t)\, w(t)\, dt = \lambda\, w(s) \ .$$

and a normalisation relation:

(4) $$\int_a^b w(t)\, dt = 1 \ .$$

The integral equation (3) can be rewritten as:

(5) $$w(s) = \int_a^b K_\lambda(s,t)\, w(t)\, dt$$

where

(6) $$K_\lambda(s,t) = \frac{J(s,t)}{\lambda} \ .$$

The equation (5) is a particular case of the typical Fredholm's integral equation:

(7) $$\alpha(s)\varphi(s) + \int_a^b K(s,t)\,\varphi(t)\, dt = f(s)$$

in which the right hand side would be the function identically null on $[a,b]$.[1] This type of equation has been studied and numerical methods of resolution have been developed and are available. It is not of our present purpose to discuss this point here.[2]

Overall Priority Function

Similarly to the discrete case, we are bound to consider a succession of dominance levels in a hierarchy. Let us first recall some notation and formulation in the discrete case in order to give the corresponding generalisation in the continuous case.

We shall remember a discrete – complete – hierarchy to be a sequence of n levels. A level i has N_i elements and therefore the i-th level priority vector P_i has N_i components and is expressed as

$$P_i = [\, p_1^i \; p_2^i \; \cdots \; p_{N_i}^i \,]^T \qquad i = 1, 2, \dots, n .$$

Between two successive levels i and $i+1$, is an eigen matrix Π_i whose columns are the eigen vectors resulting from the impact of the higher i-th level upon its successor $i+1$-st level. The matrix Π_i

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(1) As $\alpha \equiv 1$, the equation (5) is in fact a Fredholm integral equation of the second kind.

(2) For more details, see [1] and [2], and all referred and abundant literature in these books.

has dimension $N_{i+1} \times N_i$ and a recurrent relation exists between the eigen matrices and the priority vectors :

(8) $$P_i = \Pi_{i-1} \; P_{i-1} \qquad i = 1, 2, \ldots n.$$

• top objective

level 1 — N_1 elements

level 2 — N_2 elements

⋮

level n — N_n elements

Expanding the relation (8) to have the expression of a component of P_i in terms of P_{i-1}, we get:

(9) $$p_j^i = \sum_{k=1}^{N_{i-1}} \pi_{jk}^{i-1} \, p_k^{i-1} \qquad j = 1, 2, \ldots, N_i$$

From (8) and (9), expanding now the recurrence we obtain the expression of the overall priority vector P_n in terms of the first level eigen vector:

(10) $$p_i^n = \sum_{k_1=1}^{N_{n-1}} \sum_{k_2=1}^{N_{n-2}} \cdots \sum_{k_{n-1}=1}^{N_1} \pi_{i k_1}^{n-1} \, \pi_{k_1 k_2}^{n-2} \cdots \pi_{k_{n-2} k_{n-1}}^{1} \, p_{k_{n-1}}^{1}$$

An interesting remark, here, is that the expression (10) appears to be the expression of a tensor[3].

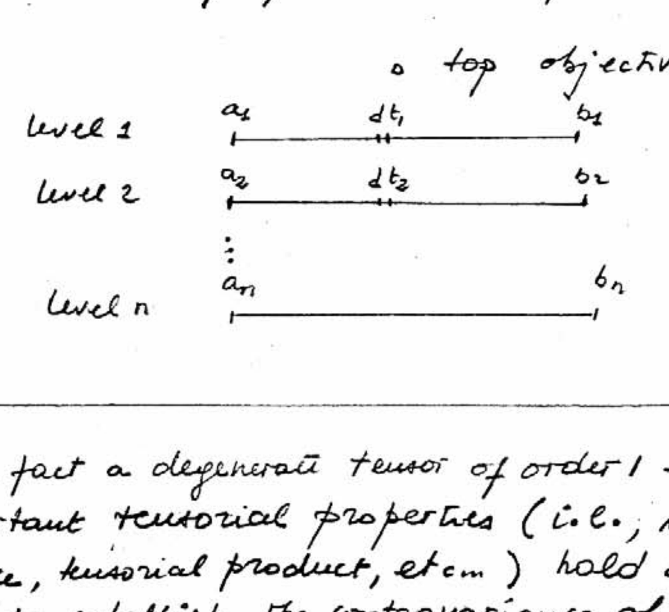
Let us now draw a generalisation to a continuous level hierarchy. This is, similarly, a succession of dominance levels, each one being a continuous interval: the i-th level is an interval $[a_i, b_i]$, say. We then can define for each level a priority function w_i taking values on $[a_i, b_i]$ and normalized over $[a_i, b_i]$; that is,

$$(11) \qquad \int_{a_i}^{b_i} w_i(t)\, dt = 1 \qquad i = 1, 2, \ldots n.$$

Now, between two successive levels, i and $i+1$, a two-argument eigen function — that we shall call the "impact function" — generalises the concept of the eigen matrices by a recurrent relation:

$$(12) \qquad w_{i+1}(x) = \int_{t=a_i}^{b_i} \pi_i(x,t)\, w_i(t)\, dt \quad , \forall x \in [a_{i+1}, b_{i+1}]$$

obviously equivalent to equation (9).



(3) It is in fact a degenerate tensor of order 1 — a hypo tensor. However, the important tensorial properties (i.e., multilinearity, covariance, tensorial product, etc...) hold and it would be of interest to establish the contravariance of the left eigenvector priority vector.

Accordingly, the expression (10) becomes the expression of the overall priority function:

$$(13) \quad \omega_n(x) = \int_{t_1=a_1}^{b_1} \int_{t_2=a_2}^{b_2} \cdots \int_{t_{n-1}=a_{n-1}}^{b_{n-1}} \pi_{n-1}(x,t_1)\, \pi_{n-2}(t_1,t_2) \ldots \pi_1(t_{n-2},t_{n-1})\, \omega_1(t_{n-1})\, dt_1 \ldots dt_{n-1}$$

for all $x \in [a_n, b_n]$.

Moreover and to be complete – if not elegant –, we can consider a general mixed hierarchy both discrete and continuous, each level of which being a sequence of finite intervals:

$$i\text{-th level} \quad \overset{a_1^i \quad b_1^i}{\longmapsto}, \overset{a_2^i \quad b_2^i}{\longmapsto}, \ldots, \overset{a_{N_i}^i \quad b_{N_i}^i}{\longmapsto}$$

and then the overall priority vector of functions (actually a function from $[0,1]^{N_1}$ to $[0,1]^{N_n}$) would be given by a combination of both (10) and (13), and despite its monstrosity (4) is easily understandable.

Consistency – The Judgment Function.

In the discrete case, the pairwise comparison matrix of judgments is clearly illustrated (5) by the idea of comparisons of stone weights. The concept of consistency then is significantly introduced and induces a very elegant and solid mathematical framework supporting the various and many applications of hierarchy and eigenvalue analysis. It is observed that, except

(4) which is our reason not to give it here explicitly!

(5) As presented in [3], page 31.

for the case of 2-dimensional matrices, real life (i.e., biased and subjective "experts" judgments due to human uncapability to grasp as a whole a complex and scattered phenomenon), provides but inconsistent matrices. Although consistency as a pure mathematical concept (as we shall see) is nicely generalised to the continuous case, it appears to be linked to continuity of judgments.

<u>Definition 1</u>: A reciprocal judgment function J defined over $[a,b]^2$ is consistent when

$$J(s,t) = J(s,u) \cdot J(u,t) \qquad \forall (s,t) \in [a,b]^2 \tag{14}$$

An alternative definition for consistency, easily deduced as equivalent to definition 1, is:

<u>Definition 2</u>: A reciprocal judgment function J defined over $[a,b]^2$ is consistent when

$$J(s,t) = \frac{w(s)}{w(t)} \qquad \forall (s,t) \in [a,b]^2 \tag{15}$$

where w is solution of equation (3).

The second is obtained from the first by noticing that the ratio $\frac{J(s,u)}{J(t,u)}$ is independent of u

and we thus can define $\frac{w(s)}{w(t)} = \frac{J(s,u)}{J(t,u)}$ to a

multiplicative constant (equal to one by normalisation). $w(s)$ thus defined obviously solves (3). The converse (i.e., Def 2 ⇒ Def 1) is trivial.

Theorem 1 : If J is positive and consistent then
$J(A,A)=1 \quad \forall A \in [a,b]$ and
$J(A,t) = \dfrac{1}{J(t,A)} \quad \forall (A,t) \in [a,b]^2$.

proof : From definition 1 $J(A,A) = J(A,A).J(A,A)$ and hence $J(A,A)=1$. Now $J(A,A) = J(A,t).J(t,A)$ implies $J(A,t) = \dfrac{1}{J(A,t)} \quad \forall A, t$.

Theorem 2 : If the reciprocal judgment function J is consistent, then $w(A) = J(A,t)$, $\forall t \in [a,b]$, is a solution of eq. (3).

This is trivially proved by substitution.

Corollary : If J is consistent, eq. (3) and (4) are simultaneously and uniquely solved by.

$$w(A) = \frac{J(A,t)}{\int_a^b J(x,t)\, dx}.$$

The following theorem gives a nice extension to theorem 5 of [3] in both its statement and its proof.

Theorem 3 : A reciprocal judgment function J defined over $[a,b]^2$ is consistent if and only if $\lambda = b-a$.

proof : $\Rightarrow$ Assume J is consistent. From equation (3) we have

$$w(A) = \int_a^b \frac{1}{\lambda} J(A,t)\, w(t)\, dt$$

Substituting under the integral sign, it comes

$$\omega(s) = \int_a^b \int_a^b \frac{1}{\lambda^2} J(s,t)\, J(t,u)\, \omega(u)\, du\, dt$$

by definition 1:

$$\omega(s) = \int_a^b \frac{1}{\lambda} \left[\int_a^b \frac{1}{\lambda} J(s,u)\, \omega(u)\, du \right] dt$$

$$\omega(s) = \int_a^b \frac{1}{\lambda}\, \omega(s)\, dt = \frac{b-a}{\lambda}\, \omega(s) \qquad \therefore \lambda = b-a.$$

$\Leftarrow$ Conversely, from the expression of λ

$$\lambda = \int_a^b J(s,t) \frac{\omega(t)}{\omega(s)}\, dt$$

taking integral of both sides:

$$\lambda(b-a) = \int_a^b \int_a^b J(s,t) \frac{\omega(t)}{\omega(s)}\, dt\, ds = \int_a^b \int_a^b G(s,t)\, dt\, ds$$

where $G(s,t) = J(s,t) \dfrac{\omega(t)}{\omega(s)}$.

$$\lambda(b-a) = \int_{s=a}^{b} \int_{t=a}^{s} G(s,t)\, dt\, ds + \int_{s=a}^{b} \int_{t=s}^{b} G(s,t)\, dt\, ds$$

Inverting the integral signs in the second integral

$$\lambda(b-a) = \int_{s=a}^{b} \int_{t=a}^{s} G(s,t)\, dt\, ds + \int_{t=a}^{b} \int_{s=a}^{t} G(s,t)\, dt\, ds$$

hence $$\lambda = \frac{1}{(b-a)} \int_{s=a}^{b} \int_{t=a}^{s} \left[G(s,t) + G(t,s) \right] dt\, ds$$

Noting that $G(s,t) = \dfrac{1}{G(t,s)} \quad \forall (s,t)$

it comes

$$\lambda = \frac{1}{b-a} \int_{s=a}^{b} \int_{t=a}^{s} \left[G(s,t) + \frac{1}{G(s,t)} \right] dt\, ds$$

But $\quad x + \dfrac{1}{x} \geq 2 \qquad \forall x \in \mathbb{R}^*$

hence, $\quad \lambda \geq \dfrac{1}{b-a} \displaystyle\int_{s=a}^{b} \int_{t=a}^{s} 2\, dt\, ds = b-a.$

λ attains its minimum value when and only when $G(s,t) + \dfrac{1}{G(s,t)}$ attains its minimum 2.

that is $\lambda = b-a \Rightarrow G(s,t) + \dfrac{1}{G(s,t)} = 2$

hence $G(s,t) = 1$ and $J(s,t) = \dfrac{w(s)}{w(t)}$

Q.E.D.

<u>Corollary</u>: For any general reciprocal judgement function over $[a,b]^2$, $\lambda \geq b-a$.

<u>Theorem 4</u>: If J is positive and consistent, and if w is a solution of eq. (3) then $J(s,t) \geq J(u,v)$ if and only if

$$\frac{w(s)}{w(t)} \geq \frac{w(u)}{w(v)} .$$

proof: by definition 2.

Examples of consistent Judgement Functions

a). $J(A,t) = e^{k(A-t)}$ $\quad A, t \in [a,b]$

for $0 < |k| \leq \dfrac{\ln \alpha}{b-a}$ where α is the judgment scale.

In this case, the eigen function is

$$\omega(A) = \frac{k e^{kA}}{e^{kb} - e^{ka}}.$$

b). Any $J(A,t) = \dfrac{f(A)}{f(t)}$ where f over $[a,b]$ is

such that $0 < f(t)$ $\quad$ ~~$\forall t$~~ $\in [a,b]$

and
$$\frac{\max\limits_{t \in [a,b]} f(t)}{\min\limits_{t \in [a,b]} f(t)} \leq \alpha$$

and
$$\frac{\min\limits_{t \in [a,b]} f(t)}{\max\limits_{t \in [a,b]} f(t)} \geq \frac{1}{\alpha}.$$

It is expected, when we give judgments, that we try to be as consistent as possible. The ideal case being attained for the most consistent. In a matrix (discrete case) of judgments, if two elements to be compared are barely differentiable with respect to a same objective, it is natural to observe, near consistency, the same or very close values of their comparisons with the other elements. So, tending to consistency and making two elements being closer and closer to each other will make a consistent judgment matrix tend toward a limiting continuous judgment function. In other words, consistency for a judgment function will correspond to continuity of this function.

Nothing prevents us from being more accurate in our definitions of consistency and we thus can define:

1. Consistency at a point : J is consistent at $t_0 \in [a,b]$ when $$J(A,t_0)\cdot J(t_0,u) = J(A,u) \quad \forall (A,u)$$

2. Consistency : J is consistent on $[a,b]$ when $$J(A,t)\cdot J(t,u) = J(A,u) \quad \forall (A,t,u)$$

We then could conjecture the following theorem:

Theorem : a reciprocal positive judgment function J is consistent at t_0 if and only if J is continuous at t_0.

J, although of two arguments, being reciprocal, has the property that it is continuous with respect to one argument if and only if it is continuous with respect to the other one. This comes as a straight forward remark from (1) and (2), and hence we can simply state "J continuous at t_0" without specifying with argument.

Conclusion

The idea of this paper has no other meaning than pure theoritical generalisation. No application to any "real life" (made-up or not) problem has been made yet, but it is not outrageous to think of an example where judgments — non-consistent if non-continuous —, could be expressed by a judgment function and then an analysis being drawn would give a priority/distribution at any point which is of real interest for computational purposes.

- REFERENCES -

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[4] T.L. Saaty, "A Scaling Method for Priorities in Hierarchical Structures", J. of Math. Psychol., Vol. 15, No. 3, June 1977, pp. 234-281.